THE ELASTIC BEHAVIOUR OF A LATERALLY LOADED PILE

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by

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A B S T R A C T

As the first step of a thorough investigation into the analysis and design of laterally loaded pile-soil systems, the ultimate aim of which is to enable the accurate prediction of behaviour from simple soils tests, this thesis presents solutions for a laterally loaded flexible pile in an elastic soil. The analysis differs from most previous attempts in that the soil is considered to be continuous.

A solution is presented, in the form of a computer program, for a single free or fixed-head pile loaded with moment and/or shear at or above ground level. The soil is taken to be linearly elastic and continuous with properties which may vary with depth, in layers.

The effects of the various pile and soil properties on system behaviour are studied, and tables are presented which give approximate ground level deflections for a pile in a soil of constant elastic properties. A pile characteristic length is defined and shown to be significant economically. Both the tables and the computer program can be used in design as long as the limitations of the elastic soil model are recognised and strength requirements are met.

Photoelastic and full-scale pile tests are described and compared with theory.

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Notation is defined where it first appears in the text. Only the more important symbols are listed below.

A	area
a	distance from centroid to neutral axis
c	constant
d_1	ground level displacement
d_2	pile tip displacement
E	Young's modulus, of pile unless obviously otherwise
e	strain
f	photoelastic constant
G	shear modulus, of soil unless obviously otherwise
G_i	shear modulus of soil layer i
h	length of pile above ground
I	moment of inertia (of pile)
i	integer subscript defining (1) direction, or (2) layer, or (3) position of point
j	similar to i
k	constant; or subscript giving direction.
k_h	horizontal modulus of subgrade reaction, here defined as the ratio of load per unit length to displacement at any point.
L	embedded length of pile
L_c	characteristic length of pile
M	applied moment
M_F	fixing moment
m	number of point forces across pile

mn	total number of point forces considered
n	number of point forces along pile; or photoelastic fringe order
$\mathcal{P}$	force vector
P_i	force in direction i ; or horizontal force at position i
p_1, p_2, p_3	principal stresses
$\mathcal{S}$	main sub-matrix in equation (3.6)
t	thickness
$\mathcal{U}$	displacement vector
u_i	displacement in direction i; or horizontal displacement at position i.
u_{ij}	u_i due to P_j
V	applied shear load; volume
W	pile width
$\underline{x}$	position vector
x_i	component of position vector
$\underline{y}$	position vector of point force
y_i	component of position vector
$\underline{z}$	dummy position vector
z_i	component of position vector; or depth of point i
Δ	top displacement of free-head pile (usually ground-level displacement)
Δ_V	top displacement of free-head pile with shear load only.
Δ_M	top displacement of free-head pile with moment load only.

Δ_F top displacement of fixed-head pile

θ top slope of free-head pile

θ_V top slope with shear load only

θ_M top slope with moment load only

μ Poisson's ratio of soil

μ_i Poisson's ratio of soil layer i

σ_{ij} stress in direction i on plane perpendicular to direction j.

CHAPTER ONE

INTRODUCTION

1.1 GENERAL

Static or dynamic lateral loads on structures are often encountered: earthquakes, wind and waves all provide lateral loading; retaining walls and bridge abutments are subject to lateral earth pressure; and wharf structures must be designed to resist the impact of a ship. Such loads must obviously be resisted by the structure foundation. In addition, some vertically loaded structures require the foundation to develop a significant horizontal reaction.

In some instances, the structure will be pile supported, and the problem of designing a laterally loaded pile, or group of piles, will arise.

This design is made difficult by the nature of soil, which is in general non-uniform, inelastic and anisotropic and which gives a non-linear response to loading. Properties may also be time-dependent. Obviously, successful analysis and design will depend largely on the way in which the soil is represented: the first step must be to choose a soil model.

1.2 SOIL MODEL

Those methods of analysing a laterally loaded pile which assume it to be rigid and to rotate about some point on its length will be disregarded for the moment, as our concern is rather with the general flexible pile.

As is shown in the next chapter, virtually all previous attempts at the analysis of a laterally loaded pile have assumed that the soil may be represented by what is known as the Winkler model, or a modification of it.

1.2.1 The Winkler Model

This model, proposed by Winkler in 1867[1], is based on the assumption that the deflection at a point is directly proportional to the force at the point and independent of forces at other points. Implicit is the property that deformation occurs only where loading exists.

The model is usually thought of as a line of closely spaced unconnected linearly elastic springs. It has generally been used as the foundation in the problem of a beam on an elastic foundation, and gives rise to the equation

$$EI \frac{d^4y}{dx^4} = p - k_o by ,$$

where
- EI = beam stiffness
- y = deflection
- x = distance along beam
- p = applied loading
- b = beam width

and k_o, or more commonly $k = k_o\, b$, is called the foundation modulus or modulus of subgrade reaction. The equation has been solved for a variety of loading and boundary conditions.[2]

An obvious treatment of the laterally loaded pile has been to consider it as a vertical beam on an elastic foundation, k being replaced by the horizontal modulus of subgrade reaction k_h. The main modification made has been to allow k_h to vary with depth: $k_h = k_h\,(x)$.

One obvious drawback to the Winkler soil model is that it assumes a linear load-deflection relationship; another is that it assumes the soil to have no continuity where real soil has at least a partially continuous nature. These unrealities have led Terzaghi[3] to say (p.345):

> "This is a highly artificial concept. Therefore the results of computation based on this concept must be considered crude estimates."

Nevertheless, the use of the Winkler model, modified in a number of ways, has continued. The usual justification has been that it provides more simple analyses, but this reasoning has become faulty on two counts. Firstly, increasing modification has led to greatly decreased simplicity; and secondly, the introduction of high speed computers has allowed computations which may previously have been regarded as impossibly lengthy to be carried out with ease. Once a computer program has been written, the simplicity of an alternative method is no longer a sufficient reason for its use.

Some limitations of the Winkler model have been mentioned, but they do not directly imply that a better model should be sought. A perfectly good computational device has been provided as long as the value of k_h at any point can be found. Unfortunately the unrealistic nature of the model causes it to be most difficult if not impossible to assign a reasonably accurate value to the foundation modulus. Terzaghi[3] may be quoted again (p.354):

> "Since [k_h] depends on many factors other than the nature of the soil it cannot be determined directly by laboratory or by small-scale field tests."

Or once again ([4] p.300):

> "... the erroneous conception [is] widespread among engineers that the numerical value of the coefficient of subgrade reaction depends exclusively on the nature of the subgrade. In other words, it [is] customary to assume that this coefficient has a definite value for any given subgrade."

Because soil is continuous, the displacement at a point, and hence k_h, will be affected by forces at surrounding points. In the case of a laterally loaded pile, this means that k_h will depend not only on the soil properties but also on the pile properties and loading.

This fact is well illustrated in later chapters.

As the ultimate aim must be to predict the behaviour of a laterally loaded pile from real soil properties, the Winkler model must be abandoned in favour of one which more realistically represents real soil.

1.2.2 Soil as an elastic continuum

The fundamental thesis governing this investigation is now presented: it is that the best initial approach to the problem of a laterally loaded pile is to treat the soil as an elastic continuum. It should immediately be emphasised that this representation will in most cases be only an initial approximation. However, there are a number of important advantages which it holds over the Winkler model.

Use of the elastic continuum allows recognition of the obvious fact that what actually happens in the soil is important. It allows the determination of stresses, a knowledge of which is probably essential for some extensions to the theory. For example, other than crude modifications to linear elastic behaviour could perhaps not be made without a knowledge of stresses. The ability to superimpose effects is useful: the study of pile groups would seem initially to require elastic stress distributions which could be superimposed.

A very important advantage is that the soil is now described by its elastic constants, real quantities which should not be too difficult to determine.

Even although real soil is not completely continuous, a sensitivity analysis can now be carried out. The relative importance of variables such as the pile width or Poisson's ratio for the soil can be studied, and a good overall understanding of system behaviour acquired.

The treatment of soil as an elastic continuum is made possible through the solutions presented by Mindlin in 1936[5]. He gave expressions, reproduced in Appendix I, for the stresses and displacement at any point in an elastic half-space caused by a force at some other point. The widely used Boussinesq equations are a special case of Mindlin's solutions and provide a precedent for the use of the elastic half-space model.

Spillers and Stoll[6] (1964) and Poulos[7] (Oct. 1968) appear to be the only previous researchers to have directly applied Mindlin's equations to the problem of a laterally loaded flexible pile. Comparisons with their solutions are made in a later chapter.

It should be noted that the term "soil" should hereafter be taken to mean "soil model" rather than "real soil".

1.3 SCOPE OF RESEARCH

The general laterally loaded pile problem has many variables. For example: loading may be static (short or long term) or dynamic; piles may be single or in groups and fixed or free at their head; soils are of many types and can have varying properties at any site; and the designer may be interested in the service load deflections and stresses or in the ultimate behaviour of the system.

The broad aim of this investigation has been to provide a sound base for a continuing research project in which as many as possible of the above problems may be investigated. The base provided is a solution for a single pile in an elastic isotropic soil. Moment and/or shear loading can be applied at or above ground level and the pile head can be free or fixed against rotation. The simple soil model has been extended so that properties can vary with position: an analysis is developed which allows a general variation, and an approximate solution is given which allows a variation with depth in a layered manner.

Besides providing an extended base for further research, the investigation has produced some results of more immediate interest.

The approximate solution mentioned above has been presented in the form of a program for the electronic digital computer which gives detailed information about a loaded pile in a layered soil. Providing the limitations of the soil model are borne in mind, and providing soil elastic constants can be found with reasonable accuracy, this program can be used in the design process.

In addition, tables have been compiled which give approximate values for the ground level displacement and (in the case of a free head pile) slope for a pile in a soil of constant properties. With the same provisions as above, these also may be used as a design tool, either directly or as a step prior to the use of the computer.

Two pile tests have been carried out, one using a small-scale photoelastic model and the other being a full-scale test. The latter has provided some interesting results.

A large proportion of the time spent on this investigation has been devoted to the development of computer programs.

CHAPTER TWO

REVIEW OF LITERATURE

The object of this review is to briefly trace major developments in the analysis of laterally loaded piles, especially of the single pile under working load, in order to provide a background to the work which follows. An annotated bibliography was compiled by Chang[8] in 1960 and an extensive bibliography was given by Davisson and Prakash[9] in 1963. The latter article is especially concerned with rigid piles (poles) but includes tabulated information concerning both test results and theoretical analyses of flexible piles as well. Broms[10,11,12] in 1964 and 1965 has also given bibliographies on laterally loaded piles, and the related problem of the beam on an elastic foundation was reviewed by Hetenyi[13] in 1966. The intention is not to duplicate the material in these papers but to outline only important aspects of the literature already covered and to mention significant recent contributions. As was mentioned in Chapter 1, the bulk of previous work on the problem of a laterally loaded flexible pile has used the Winkler soil model or a modification of it, so a brief review to give a background is all that is desirable in this thesis.

The first article referring specifically to laterally loaded piles appears to have been a description of pile tests which appeared in 1880[14]. A series of tests carried out by the U.S. Corps of Engineers was reported in 1935 by Feagin[15] and subsequent discussions by Cummings[16] and Chang[17] introduced theoretical analyses based on the Winkler assumptions. Chang assumed a uniform soil modulus and Cummings assumed a linear variation with depth, but used several simplifying assumptions. Most previous authors, for example Raes[18], had considered the pile to be rigid and to rotate about some point, often the pile tip.

A significant step was made with the application of numerical methods to the solution of the basic pile equation. Palmer and Thompson[19] in 1948 used a difference equation method to solve the equation

$$EI \frac{d^4y}{dx^4} = -k \left(\frac{x}{L}\right)^n y$$

where L is the pile length, and k and n are constants describing the variation of soil modulus with depth. The American Society for Testing Materials in 1953 sponsored a symposium on lateral load tests on piles and this brought forward both test reports and some further difference equation solutions, in particular by Gleser[20], Mason and Bishop[21], and Palmer and Brown[22].

The general form of variation of soil modulus given above seems to have been introduced by Miche[23] in 1930. It has been often used since, although most authors have assumed that n = o for cohesive and n = 1 for cohesionless soils. Some analytic solutions of other variations have been given, for example by Reddy and Valsangkar[24], who gave a power series solution for the polynomial variation $(k_o + k_1 x^2)$. One of the main advantages of numerical method solutions, of course, is that any variation of soil modulus can be considered. Davisson and Gill[25] considered a layered soil system.

Variation of the soil modulus with displacement has also been considered. Matlock and Reese[26] obtained a solution assuming the modulus to vary with depth in some manner and then successively adjusted the form of variation to make deflections and reactions conform with a previously determined load-deflection relationship. Kubo[27] postulated a general form of variation with both depth x and displacement y given by the term $kx^m y^n$ and used experiments in sand to conclude that m = 1.0 and n = 0.5.

The soil modulus was now able to vary with both depth and deflection, but two problems remained. The first was that analyses were rather complex for hand calculation by practising engineers. Czerniak[28] and Broms[12] have both given design techniques, but the former considers only the strength analysis of a stiff pile and the latter shows poor correlation with test results. A second more fundamental problem was that of determining realistic values for the horizontal modulus of subgrade reaction. Some writers recognise the problem: Yang[29] said

> "width and elastic stiffness of pile are very significant on the evaluation of horizontal subgrade reaction. A particular value . . . should be developed for each type of pile." ;

and Matlock and Reese[26] noted

> "This modulus is essentially only a computation device . . . It is not a unique soil property." .

Generally the assumption has been that a plate bearing test will give k_h, although Kubo[27] has suggested standard penetration tests, and McClelland and Focht[30] used a triaxial test. Terzaghi[4] has given broad ranges within which the soil modulus should lie, and Vesic[31] has related k to the Young's modulus for an infinite beam on an elastic foundation. Also, use of a test pile has been made to predict the distribution of k_h at a site. For example, Gleser[20] used the form $k\left(\frac{x}{L}\right)^n$, obtained k and n from a test on a free-head timber pile, and found reasonable correlation when the same values were applied to a fixed-head pile at the same site. Agarwal and Prakash[32] used a test on a single pile to predict the behaviour of pile groups. Obviously, however, it is not a satisfactory situation if laterally loaded pile design must always include a pile test, so the problem of finding k_h remains.

It was thus a significant step when, in 1964, Spillers and Stoll[6] introduced the elastic half-space as a soil model, even although their analysis contains an error. They have given no way of finding the soil shear modulus G or Poisson's ratio μ, but the significant fact is that these properties are characteristic of the soil alone: they do not depend on pile properties or on the loading conditions. Spillers and Stoll also introduced, in an elementary way, some plasticity near the soil surface. Their analysis is discussed more fully in Chapter 3.

So far, this chapter has been concerned mostly with the deflection analysis of a single vertical pile, since that is the concern of this thesis. Some other aspects of laterally loaded pile analysis and design will now be mentioned.

Pile groups have been considered by Francis[33], but he does not account for the interaction effects which will be present in a closely spaced group of piles. Many pile tests or experiments (for example by Wen[34]) have considered pile groups, the groups often containing battered (raker) piles. In connection with such groups it is worth remembering the fact, noted by Norman[35], that a group of vertical piles will provide a better response to earthquake loading, because of the greater flexibility.

For strength analyses, the papers by Czerniak[28] and Broms[12] may be mentioned again. The method of Blum[36], also given in the Peine Pile Handbook[37], was shown by Carioti, Elms and Peace[38] to give good agreement with a full-scale pile test.

Finally, some mention should be made of "improved"

Winkler soil models, which have mostly been applied to the beam on elastic foundation problem. These models, reviewed by Kerr[39], generally aim at introducing viscoelastic behaviour and/or continuity into the foundation. They are subject, however, even more obviously to the limitations of the simple Winkler model: because they are artificial soil models it is difficult to obtain properties. This has been well illustrated by Barden[40], who notes that a string joining the tops of the springs would provide the high reaction forces which occur at the edges of a loaded rigid stamp. However, the value assigned to the string tension must be quite arbitrary, and one might just as well have arbitrarily chosen values for point forces at the edges.

CHAPTER THREE

THE ANALYSIS OF A LATERALLY LOADED PILE IN A UNIFORM ELASTIC HALF-SPACE

3.1 INTRODUCTION

The thesis has already been presented that the best first step in solving a laterally loaded pile-soil system is to consider the soil as elastic and continuous. This chapter presents a method for the solution of a single pile in a linearly elastic homogeneous isotropic half-space. The solution, as has been mentioned, is made possible by the Mindlin equations for a point force in a half-space and has already been carried out by Spillers and Stoll[6] and more recently by Poulos[7].

The solution developed differs from each of these two and comparisons are made at the end of the chapter.

The pile head may be free or fixed and may be at the ground surface or above.

The distribution of soil reaction across the pile width is non-uniform.

3.2 FREE HEAD PILE

3.2.1 Introduction

The system to be solved is shown in fig. 3.1, in which some notation is introduced. Position actions are defined as those producing "positive" pile curvature, concave

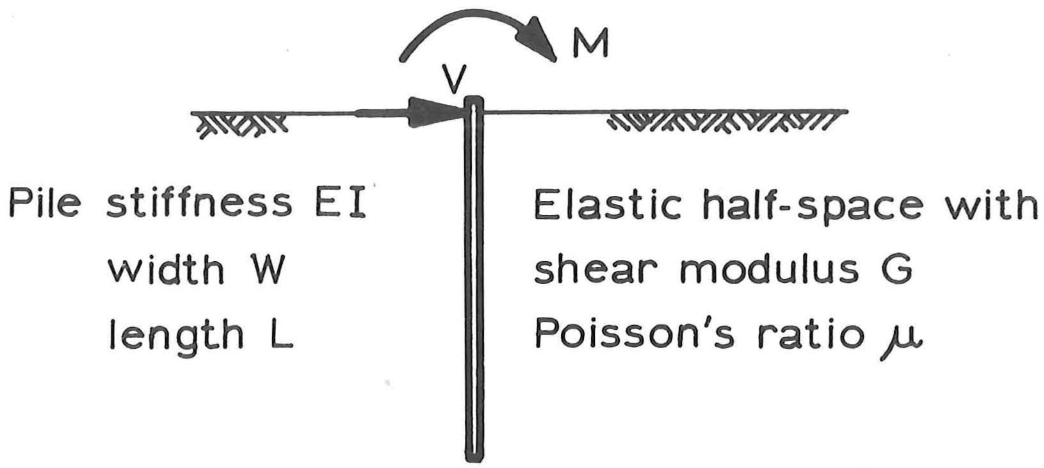


FIG. 3.1 - SYSTEM TO BE SOLVED

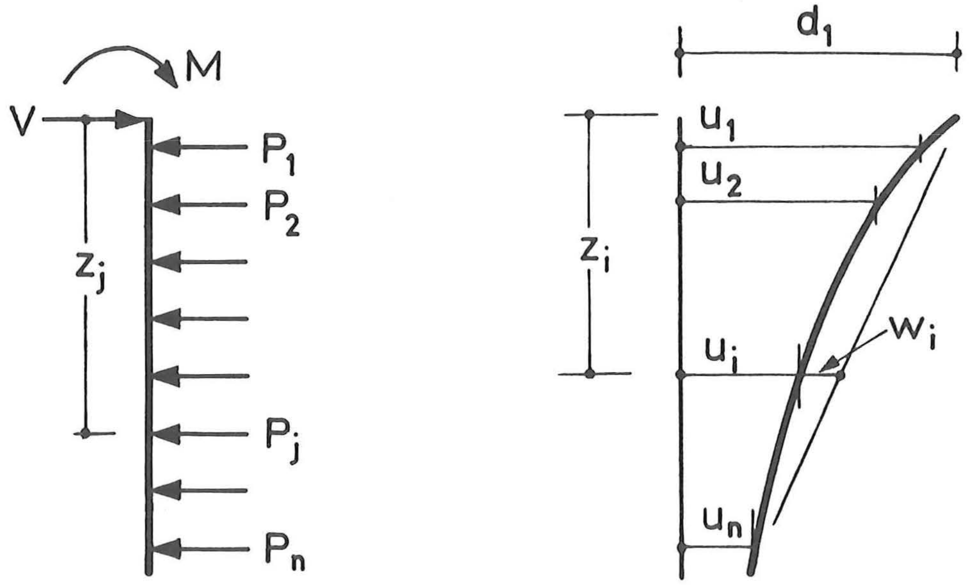


FIG. 3.2 - ACTIONS ON AND DISPLACED SHAPE OF PILE.

to the right. Thus the applied actions in fig 3.1 are positive, soil reaction is positive if acting to the left, pile action on the soil is positive to the right, and this will produce positive displacements, to the right.

Initially the soil reaction will be assumed to consist of n equally spaced horizontal point forces acting at the centres of pile segments of length L/n, refinements being introduced later. Pile displacements will be calculated using the beam equation, soil displacements using the Mindlin equations, and compatibility of forces and displacements will be preserved.

3.2.2 Pile Equations

Additional notation to be used is given in fig. 3.2 which shows the actions on and the displaced shape of the pile. There are (2n + 2) unknowns: the n soil reaction forces and the (n + 2) pile displacements.

w_i is positive for positive curvature, so

$$u_i = \frac{L-z_i}{L} d_1 + \frac{z_i}{L} d_2 - w_i$$

Restricting ourselves to a uniform beam, and neglecting shear deformation, we may write

$$w_i = \frac{Mz_i}{6EIL}(2L-z_i)(L-z_i) - \sum_{j=1}^{n} \frac{P_j z_i (L-z_j)}{6EIL}(z_i^2+z_j^2-2Lz_j)$$

$$+ \left[\frac{P_j z_j}{6EIL}(z_i-z_j)^3 \right]_{z_i > z_j} \quad \text{only}$$

That is, $w_i = a_i M + b_{ij} P_j$

[We are using the common convention that a repeated subscript implies a summation of terms over all values of the subscript: thus

$$b_{ij} P_j \text{ means } (b_{i1} P_1 + b_{i2} P_2 + \ldots b_{in} P_n)$$]

The n equations for pile displacement may now be written:

$$u_i = \frac{L-z_i}{L} d_1 + \frac{z_i}{L} d_2 - a_i M - b_{ij} P_j \qquad (3.1)$$

3.2.3 Soil Equations

It is required for compatibility that the soil displacements are equal to the pile displacements, so they may be written as u_i. Similarly, equilibrium considerations allow the forces on the soil to be written as P_j.

Mindlin has expressed the horizontal displacement u at (x,y,z) in an elastic half-space due to a horizontal force P at (0,0,c) by the equation

$$u = \frac{P}{16\pi G(1-\mu)} \left[\frac{3-4\mu}{R_1} + \frac{1}{R_2} + \frac{x^2}{R_1^3} + \frac{(3-4\mu)x^2}{R_2^3} + \frac{2cz}{R_2^3}\left(1 - \frac{3x^2}{R_2^2}\right) + \frac{4(1-\mu)(1-2\mu)}{R_2 + z + c}\left(1 - \frac{x^2}{R_2(R_2 + z + c)}\right)\right] \qquad (3.2)$$

where $R_1 = (x^2 + y^2 + (z-c)^2)^{\frac{1}{2}}$

and $R_2 = (x^2 + y^2 + (z+c)^2)^{\frac{1}{2}}$

The equation may be considerably simplified if x = y = o to (changing notation)

$$u_{ij} = \frac{P_j}{16\pi G(1-\mu)}\left[\frac{3-4\mu}{|z_i-z_j|} + \frac{1+2(1-\mu)(1-2\mu)}{z_i + z_j} + \frac{2z_i z_j}{(z_i + z_j)^3}\right] \quad (3.2a)$$

where u_{ij} is that part of u_i due to the single force P_j.

It will be noticed that if $i = j$ the first term inside the bracket becomes infinite. The method of dealing with this singularity will be discussed later. Meanwhile, the n soil equations may be written as

$$u_i = c_{ij}\ P_j \quad (3.3)$$

3.2.4 Equilibrium Equations

At present there are 2n equations in (2n + 2) unknowns. A further two equations may be obtained by considering pile equilibrium:

$$\left.\begin{aligned} \sum_{j=1}^{n} P_j &= V \\ \text{and} \qquad P_j z_j &= -M \end{aligned}\right\} \quad (3.4)$$

3.2.5 Solution of Equations

Equations (3.1) and (3.3) may be combined to eliminate u_i, giving the n equations

$$\frac{L-z_i}{L} d_1 + \frac{z_i}{L} d_2 - a_i M - b_{ij} P_j - c_{ij} P_j = 0$$

$$\text{or} \quad r_{ik}\, d_k + s_{ij}\, P_j = a_i M \quad (3.5)$$

(where j takes the values 1 - n, and k takes the values 1,2).

These may now be combined with (3.4) in matrix form to give

$$\left[\begin{array}{cc|cc} S_{ij} & \cdots & \vdots & \\ & \cdots & r_{ik} & \\ & & \vdots & \\ \hline \cdots & 1 \cdots & 0 & 0 \\ \cdots & z_j \cdots & 0 & 0 \end{array}\right] \left\{\begin{array}{c} P_j \\ \hline d_k \end{array}\right\} = \left\{\begin{array}{c} a_i M \\ \hline V \\ -M \end{array}\right\}$$

or

$$\left[\begin{array}{c|c} S & R \\ \hline T & \text{ZERO} \end{array}\right] \left\{\begin{array}{c} P \\ \hline D \end{array}\right\} = \left\{\begin{array}{c} A \\ \hline B \end{array}\right\} \qquad (3.6)$$

whence

$$D = (TS^{-1}R)^{-1}(TS^{-1}A - B)$$

and

$$P = S^{-1}(A - RD)$$

The displacements can now be found from equations (3.1) or (3.3).

3.2.6 Refinements

It is not intended at this point to examine the reality of the soil model used. It is necessary though to question whether the theory outlined is adequate for a pile in an elastic half-space, and this involves examining the approximation of the distributed soil reaction by a line of point forces.

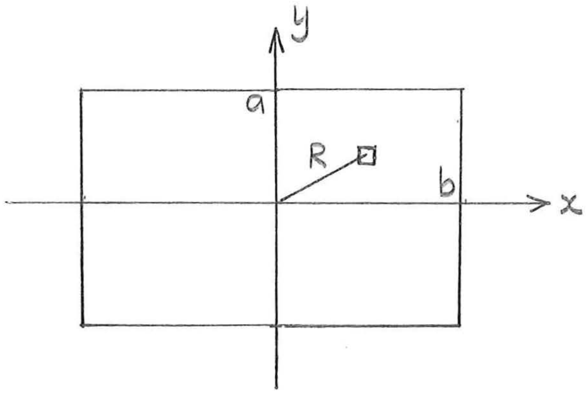
Considering the pile, this assumption is quite adequate. It does not matter how the reaction is distributed across the width since the pile is reasonably considered as a beam. Along the length, a measure of the accuracy may be obtained by comparing with a simply-supported beam under a uniformly distributed load: if the load is approximated by seven point forces, the centre deflection is less than 1% high. In fact, n will be of the order of 20 or 30, so errors will be negligible.

Considering the soil, however, it becomes obvious that some improvements will be necessary. The first concerns the singularity mentioned previously, which occurs because of the physically impossible mathematical concept of a force acting on an infinitely small area and so producing an infinite displacement at that point. Spillers and Stoll replaced the term $|z_i - z_j|$ in (3.2a), now zero, by one half the radius of the circle with area WL/n. That is, they assumed the average distance of the distributed force from the centre of the rectangular pile element to be equal to $\frac{1}{2}\sqrt{\frac{WL}{n}}$. As the terms with i = j are of prime importance, it was decided to improve on this assumption by subdividing the area W x L/n into a grid of smaller areas. The whole rectangle still displaces as a rigid plate, but the soil reaction may be non-uniform across the pile width. Thus the on-diagonal terms in the flexibility matrix $\underset{\sim}{S}$ are determined from a computer subroutine which considers each rectangle in turn to be a rigid plate with about 50 point forces acting on it. Equation (3.2) must be used instead of the simplified form because y is not always zero.

The problem has now been minimized but not eliminated, for within this subroutine it is still necessary to evaluate the displacement at a point due to a force at the same point: the validity of Spillers and Stoll's assumption must still be examined.

The examination is best made by independently deriving a replacement expression for R_1 in equation (3.2) when this term becomes zero.

The situation is shown below: the small rectangular area with sides 2a and 2b has a point force acting at its centre, and it is necessary to find the displacement at this point.



The point force Q may be represented by a uniformly distributed load acting on the rectangle.

As a and b are very small compared with other lengths equation (3.2) may be written as $u = c\frac{P}{R}$, where c is a constant. The displacement at the centre of the rectangle is thus

$$u = \frac{cQ}{ab}\int_0^a \int_0^b \frac{dx\ dy}{\sqrt{x^2+y^2}}$$

$$= \frac{cQ}{ab}\int_0^a \sinh^{-1}\left(\frac{b}{y}\right) dy$$

$$= \frac{cQ}{a}\left[\frac{a}{b}\sinh^{-1}\left(\frac{b}{a}\right) - \frac{k}{b}\sinh^{-1}\left(\frac{b}{k}\right) + 2\coth^{-1}\exp\left(\sinh^{-1}\left(\frac{b}{a}\right)\right)\right.$$

$$\left. -2\coth^{-1}\exp\left(\sinh^{-1}\left(\frac{b}{k}\right)\right)\right]_{k\to 0}$$

$$= cQ\left[\frac{1}{b}\sinh^{-1}\left(\frac{b}{a}\right) + \frac{1}{a}\sinh^{-1}\left(\frac{a}{b}\right)\right] .$$

If $a = b$,

$$u = 1.76\ \frac{cQ}{\sqrt{ab}}$$

If $a = 2b$,

$$u = 1.70\ \frac{cQ}{\sqrt{ab}}$$

The grid is made as close as possible to square and will, except in exceptional cases, be close enough that $.8a < b < 1.2a$, so the value

$$u = 1.75\ \frac{cQ}{\sqrt{ab}}$$

is always used.

In fact the grid is small enough that this term is not greatly significant: a solution with

$$u = 1.25\ \frac{cQ}{\sqrt{ab}}$$

gave a change in pile top displacement of less than $\frac{1}{2}$%.

If the grid has point spacings $r = 2a$ and $s = 2b$, then

$$u = 3.5 \frac{cQ}{\sqrt{rs}}$$

and an equivalent value for R_1 in equation (3.2) is

$$R_1 = .285 \sqrt{rs}$$

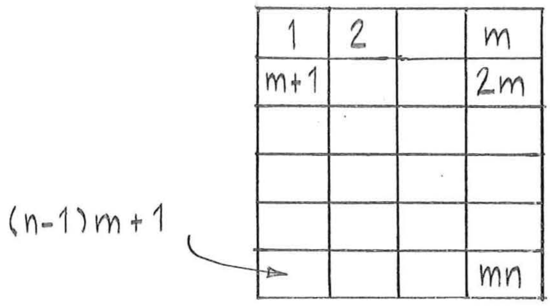
It is interesting that for a square grid, the substitution of Spillers and Stoll gives

$$R_1 = .283 \sqrt{rs}$$

which shows good agreement. It should be noted, however, that the derivation given here has assumed both a small point spacing and a near-square grid: neither condition is necessarily met by Spillers and Stoll.

Having calculated the on-diagonal terms in matrix S allowing the force distribution across the pile to be non-uniform, it now appears desirable to generally consider more than one point force across the pile. Kubo[27] has shown that the effects of pile width are important and non-linear, and one of the advantages of using a continuum approach is that these effects can be considered. Also, a very obvious way of causing point forces to approximate more closely to a distributed load is to use a small grid of forces. As a near-square grid is desirable, it becomes necessary to consider more than one force across the pile.

The pile is thus divided into m vertical strips as well as n horizontal ones. There are now mn forces acting instead of n, so there are (m - 1)n new unknown forces. But there are the further conditions that $u_1 = u_2 = \ldots u_m$ across a strip, or $(m - 1)n$ more equations. The problem may be stated so that S in (3.6) remains two dimensional by numbering the forces thus:



Once again, the full form of (3.2) must be used.

The type of force distribution obtained is shown in fig. 3.3 and the effects of pile width are discussed in the next chapter.

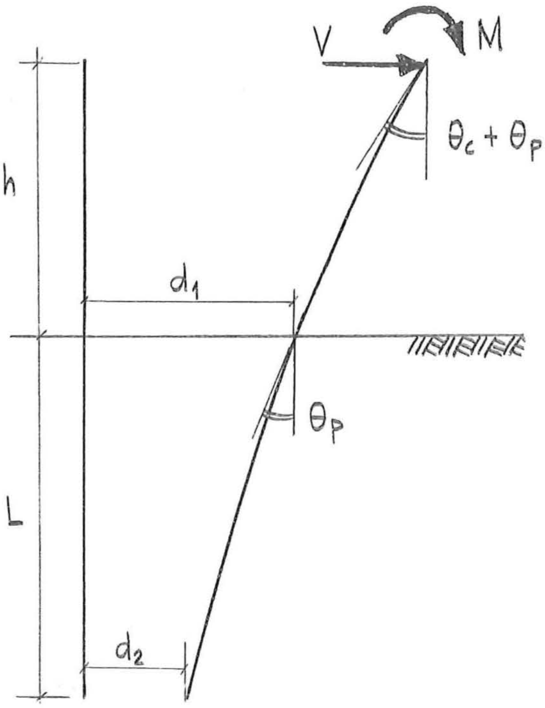
We are now able, if necessary, to represent the distributed soil reaction by a large number of point forces. The effect of the value chosen for mn is discussed in a later section.

It is easy to treat the case where applied actions act at a height h above the ground surface. The applied actions are converted to equivalent ground level ones, the system is solved (the slope at ground level also being calculated) and the cantilever length h is added to the pile, allowing the displacement and slope at load level to be obtained.

3.3 FIXED HEAD PILE

In many cases in practice, a laterally loaded pile will be fixed at its head. This case is a quite simple extension of the previous section, for it introduces one new unknown, the fixing moment, but provides the condition that the top slope is zero.

Consider



$$\theta_p = \frac{d_1 - d_2}{L} + \frac{1}{6EIL^2}\left[2ML^3 + P_j z_j (L - z_j)\left\{2(L - z_j)^2 + z_j\,(3L - 2z_j)\right\}\right]$$

and $\quad \theta_c = \frac{h}{EI}\,(M - \frac{Vh}{2})$

but $\quad \theta_c + \theta_p = 0$

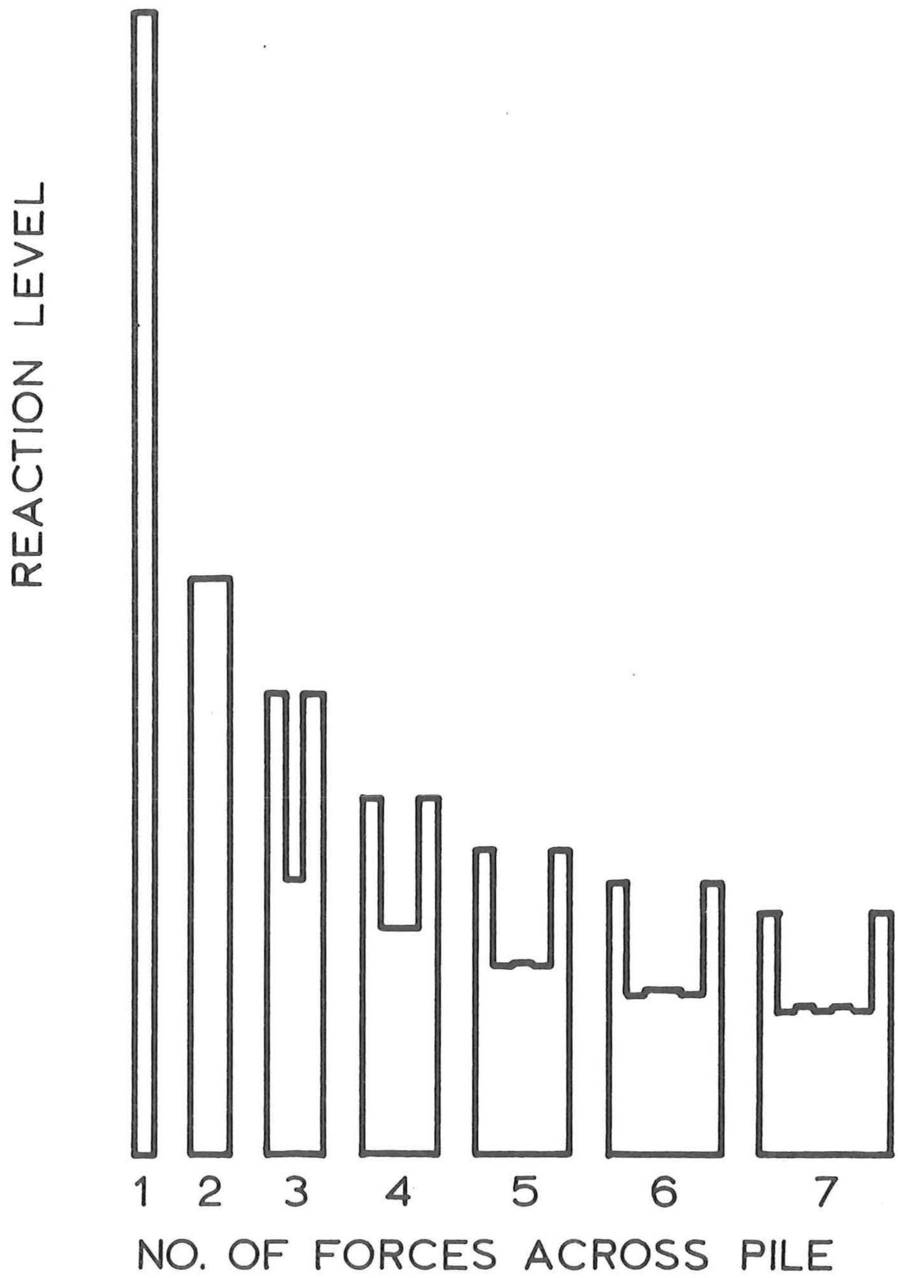


FIG. 3.3 - THE DISTRIBUTION OF SOIL REACTION ACROSS PILE AS WIDTH VARIES.

$$\text{so} \qquad ad_1 + bd_2 + cM + k_jP_j = \frac{Vh^2}{2EI} \qquad (3.7)$$

Equations (3.4) and (3.5) may now be rearranged so that the unknown M is on the left hand side and combined with (3.7) to re-express (3.6) in the form

$$\left[\begin{array}{c|c|c} S & -\frac{A}{M} & R \\ \hline k_j & c & a \quad b \\ \hline T & \vdots \\ 1 & \vdots \quad \vdots \end{array}\right] \left\{\begin{array}{c} P \\ \hline M \\ \hline D \end{array}\right\} \qquad \left\{\begin{array}{c} \vdots \\ \vdots \\ \hline Vh^2/2EI \\ \hline V \\ \vdots \end{array}\right\}$$

which may be solved in a manner similar to that previously given.

The problem of a fixed head pile could, of course, be solved by firstly solving for a free-head pile with moment only and shear only, and then calculating the moment required for zero top slope. The direct method given, however, is more efficient.

3.4 THE COMPUTER PROGRAM

The theory contained in previous sections has been used in programming the University of Canterbury's IBM 360/44 computer to obtain the solutions outlined.

The programs solving for free and fixed-head piles have been written as separate phases so either may be called by the job control statements EXEC FREE or EXEC FIXED. Input, read from cards, consists of pile and soil properties and the desired number of grid points mn. m and n are chosen internally so as to get the nearest to square grid. Printed output, a sample of which is given in fig. 3.4, consists of the input quantities and the pile deflected shape, reaction loading, bending moment and shear force distributions and (in the case of the free-head pile) the top slope. One subprogram is used, to calculate the on-diagonal terms of matrix S, as previously outlined.

Although the Winkler model is not used, sample output included in this thesis also gives values of an effective horizontal modulus of subgrade reaction k_h. This is simply a value of the ratio of reaction per unit length to displacement, and is included for the purpose of discussion. It should be remembered that when k_h is referred to subsequently, it does not imply any use of this effective modulus.

Calculation time on the 360/44 for a pile with mn = 40 is about 20 seconds, using single precision arithmetic.

Program phases FREE and FIXED are later extended to solve for a non-uniform soil model, causing the constant elastic half-space to be a special case.

PROGRAM VARIABLES	NO OF CYCLES = 1
	N = 30
	M = 1
PILE PROPERTIES	HEAD FREE
	LENGTH = 480.0
	WIDTH = 12.0
	E =0.3000E 08
	I =0.1000E 05
LOADING DETAILS	HT ABOVE GROUND = 120.0
	SHEAR =0.4000E 05
	MOMENT =0.0
SOIL PROPERTIES	1 LAYER SOIL
	G = 1000.0
	PR = .45

DEPTH	REACTION	DEFLECTION	SHEAR	MOMENT	SLOPE	KH
-120.0		0.8819	40000.0	0.	0.4545E-02	
000.0		0.3749	40000.0	4800000.		
8.0	17467.5	0.3467				3148.9
16.0			22532.5	5300259.		
24.0	13274.8	0.2937				2825.0
32.0			9257.6	5554579.		
40.0	10453.7	0.2453				2663.1
48.0			-1196.0	5619071.		
56.0	8058.1	0.2018				2496.3
64.0			-9254.1	5535469.		
72.0	6039.0	0.1629				2316.4
80.0			-15293.1	5339090.		
88.0	4340.7	0.1288				2106.8
96.0			-19633.8	5059674.		
104.0	2922.1	0.0990				1843.9
112.0			-22555.9	4722155.		
120.0	1749.5	0.0735				1487.7
128.0			-24305.4	4347263.		
136.0	793.1	0.0518				956.6
144.0			-25098.5	3952031.		
152.0	26.3	0.0337				48.8
160.0			-25124.8	3550243.		
168.0	-575.6	0.0188				-1918.2
176.0			-24549.3	3152849.		
184.0	-1035.1	0.0067				-9685.3
192.0			-23514.2	2768341.		
200.0	-1372.8	-0.0029				29901.3
208.0			-22141.4	2403096.		
216.0	-1607.6	-0.0102				9837.6
224.0			-20533.7	2061695.		
232.0	-1756.4	-0.0157				7012.4
240.0			-18777.4	1747206.		
248.0	-1834.2	-0.0195				5887.4
256.0			-16943.2	1461441.		
264.0	-1854.6	-0.0219				5287.4
272.0			-15088.6	1205186.		
280.0	-1829.2	-0.0232				4919.8
288.0			-13259.4	978402.		
296.0	-1768.1	-0.0236				4678.0
304.0			-11491.3	780397.		
312.0	-1680.0	-0.0233				4514.4
320.0			-9811.3	609977.		
328.0	-1572.2	-0.0223				4405.6
336.0			-8239.0	465574.		
344.0	-1450.7	-0.0209				4339.8
352.0			-6788.3	345356.		
360.0	-1320.3	-0.0191				4312.1
368.0			-5468.0	247305.		
376.0	-1184.7	-0.0171				4323.0
384.0			-4283.3	169295.		
392.0	-1047.0	-0.0149				4378.5
400.0			-3236.3	109138.		
408.0	-909.0	-0.0126				4493.6
416.0			-2327.4	64629.		
424.0	-772.1	-0.0103				4700.3
432.0			-1555.2	33568.		
440.0	-637.5	-0.0079				5075.0
448.0			-917.7	13784.		
456.0	-506.2	-0.0054				5843.2
464.0			-411.5	3150.		
472.0	-411.2	-0.0030				8648.0
480.0		-0.0018	-0.3	-144.		

FIG. 3.4 - TYPICAL OUTPUT FROM COMPUTER.

3.5 RESULTS, WITH COMPARISONS

3.5.1 Introduction

The aim of this section is to examine the effect of the programming variable mn, to show typical pile behaviour, and to compare this behaviour with that obtained by Spillers and Stoll[6] and by Poulos[7]. The effects produced by changing the various pile and soil properties are examined in the next chapter.

3.5.2 The Effect of Varying the Number of Point Forces Acting.

As the distributed reaction on the face of the pile has been approximated by a number of point forces, it is necessary to know how many such forces should be considered so that a reasonable standard of accuracy is preserved. Fig. 3.5 shows how the top displacement varies with mn for the pile with $EI = 10^{11}$, $L = 200$, $W = 20$, and loading $V = 10^4$ and $M = 10^6$ in the soil with $G = 1000$ and $\mu = .45$. This is a medium length pile (the term will be properly explained in Chapter 4). From the top plot, it can be seen that the points show both a general trend and a scatter. The random scatter is of the sort that can be expected because of arithmetic inaccuracies within the computer (the 360/44 working in single precision uses only six significant hexadecimal digits and errors can occur due to truncation and chopping) but the significance of the

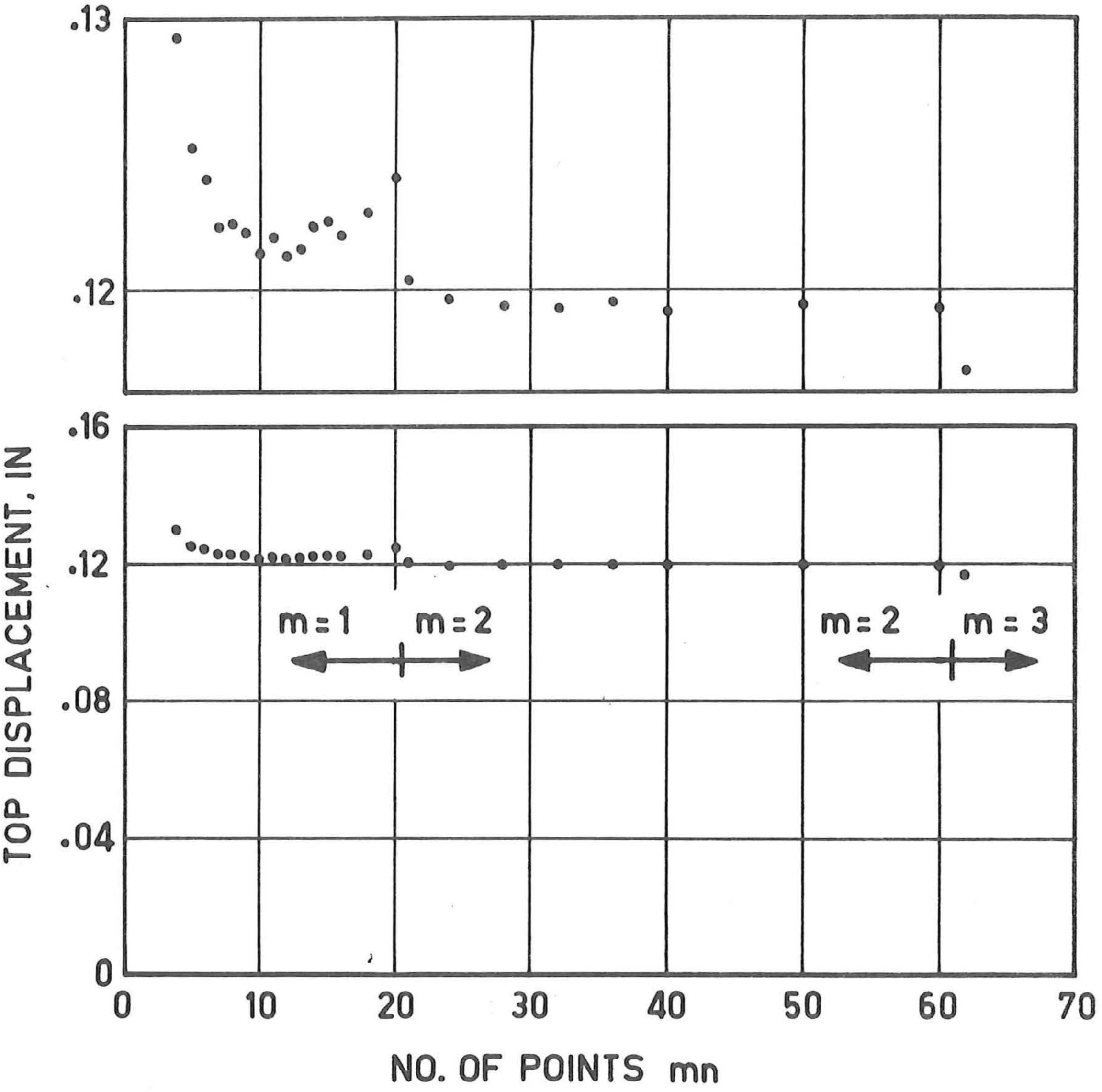


FIG. 3.5 – TOP DISPLACEMENT TO TWO SCALES AGAINST NUMBER OF GRID POINTS USED

scatter itself can be seen to be slight. The general trend is more significant and shows two facts: (1) there is a general decrease in top displacement with an increase in mn, and (2) there is a local increase in top displacement where m changes value, that is, where the grid shape is furthest from square.

The lower plot is intended to show more clearly the significance of these variations. At the time of development the University of Canterbury computer only allowed a maximum value for mn of 66 when the extended program (later described) was used. It is believed however, that this is completely adequate for practical use, giving conservative errors of very small magnitude.

3.5.3 Typical Results

We shall now return to fig. 3.4 which gives a solution for, once again, a medium length pile. A most important observation from this figure concerns the values of the effective horizontal modulus of subgrade reaction k_h. It will be seen that although the soil has constant properties its continuous nature causes k_h to vary considerably with depth. Because the displacement at a point is caused not only by the force at that point but also by all other forces, k_h is lower where there are larger "helping forces" near the point. The proximity of the free surface also affects k_h. The values shown confirm the earlier statement that k_h is a property of pile as well as soil properties, and emphasise the fact that a foundation modulus probably cannot be related to fundamental soil properties. It does not matter that a real soil is not completely continuous: a partially continuous soil will show the same effect to a lesser degree.

Penzien, Scheffey and Parmelee[41] have come to the opposite conclusion: they used the Mindlin equations and noted firstly the rapid decay of deflection with distance from a point load and secondly the relative constancy of k_h with depth when a vertical line of equal point forces acts in a half-space. They concluded that k_h may be related to soil elastic constants and they then used the Winkler model to solve for a laterally loaded pile system. By engaging in a lengthy investigation of the elastic half-space as a soil model, and by using elastic constants to find k_h, they in fact admitted the superiority of a continuous model but concluded that the superiority is not great enough to warrant the use of this model. Fig. 3.4 shows quite clearly, however, that when there exists a non-uniformly distributed load the variation of k_h with position can be very significant. It will be noted that the variation is not even gradual. At a change in displacement direction there is a jump in the value of k_h: the very different values close to this change are not significant as the displacements are very small, but at a short distance from the change k_h can be seen to have acquired a significantly different value.

Figures 3.6 and 3.7 show some typical results. Figure 3.6 gives the displaced shape and the bending moment and shear force diagrams for a relatively flexible pile, the example pile given by Spillers and Stoll ($E = 30 \times 10^6$ psi, $I = 4010 in^4$, $W = 14$ in, $L = 40$ ft, $G = 1187$ psi, $\mu = .45$, and $V = 40^k$ at 12 ft above ground level). Fig. 3.7 shows displacement profiles for a short and a long pile under shear and moment loading. It is obvious that the classification of piles into classes (short or rigid and long or flexible) may be useful. The displacement profile for a long pile is similar to a heavily damped wave form, this characteristic having been displayed by most reported experimental profiles.

3.5.4 Comparison with solution of Spillers and Stoll

A solution has been obtained for the example pile given by Spillers and Stoll[6]. Fig. 3.8 compares the two elastic solutions and shows one to be in error. No details of Spillers and Stoll's method are known, but their solution appears to be erroneous for two reasons: their shape does not seem as reasonable; and their solution definitely contains some error, because their moment and deflection curves do not correspond, the moment changing sign but the pile curvature being of one sense.

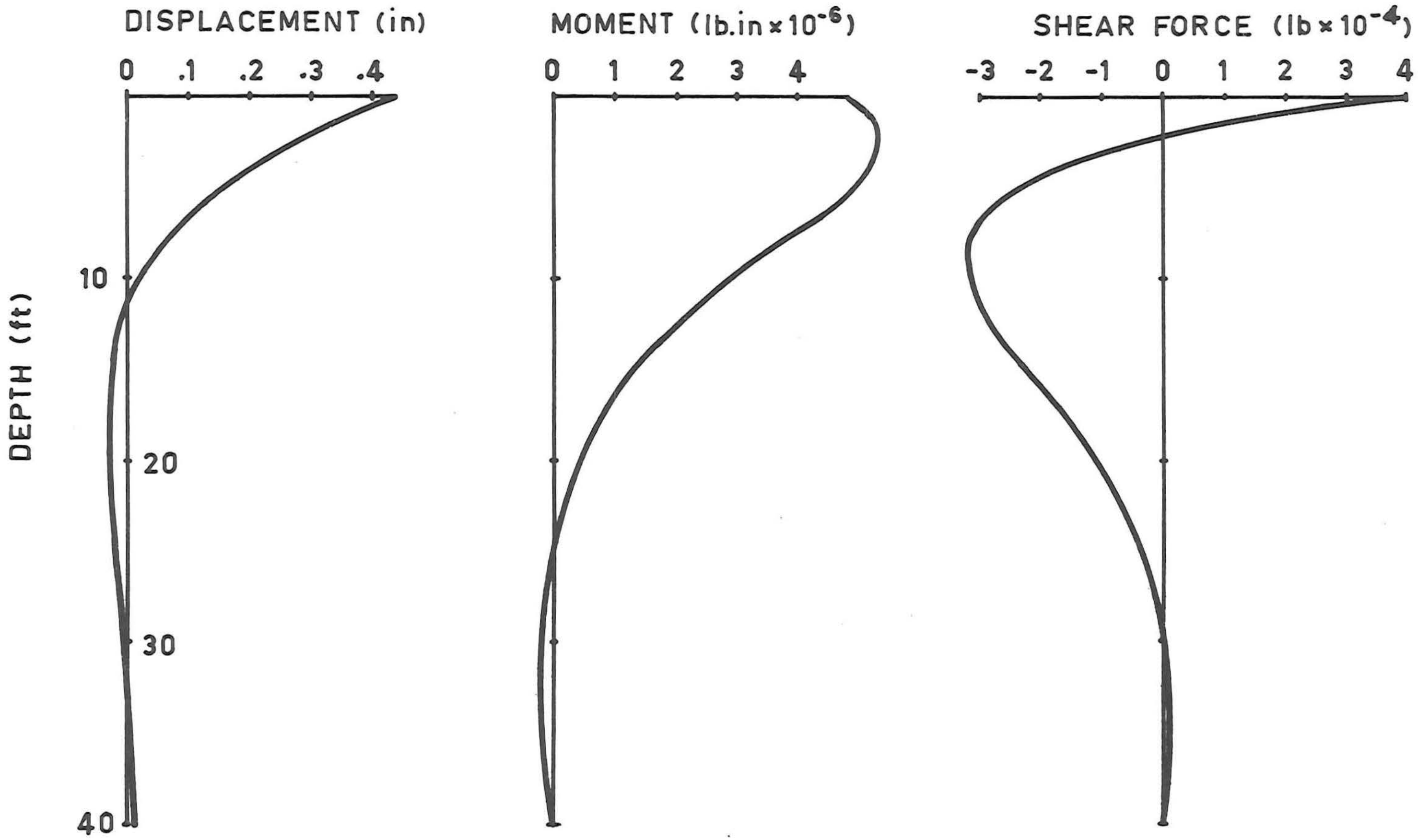


FIG. 3.6 - SOLUTION FOR A TYPICAL FLEXIBLE PILE.

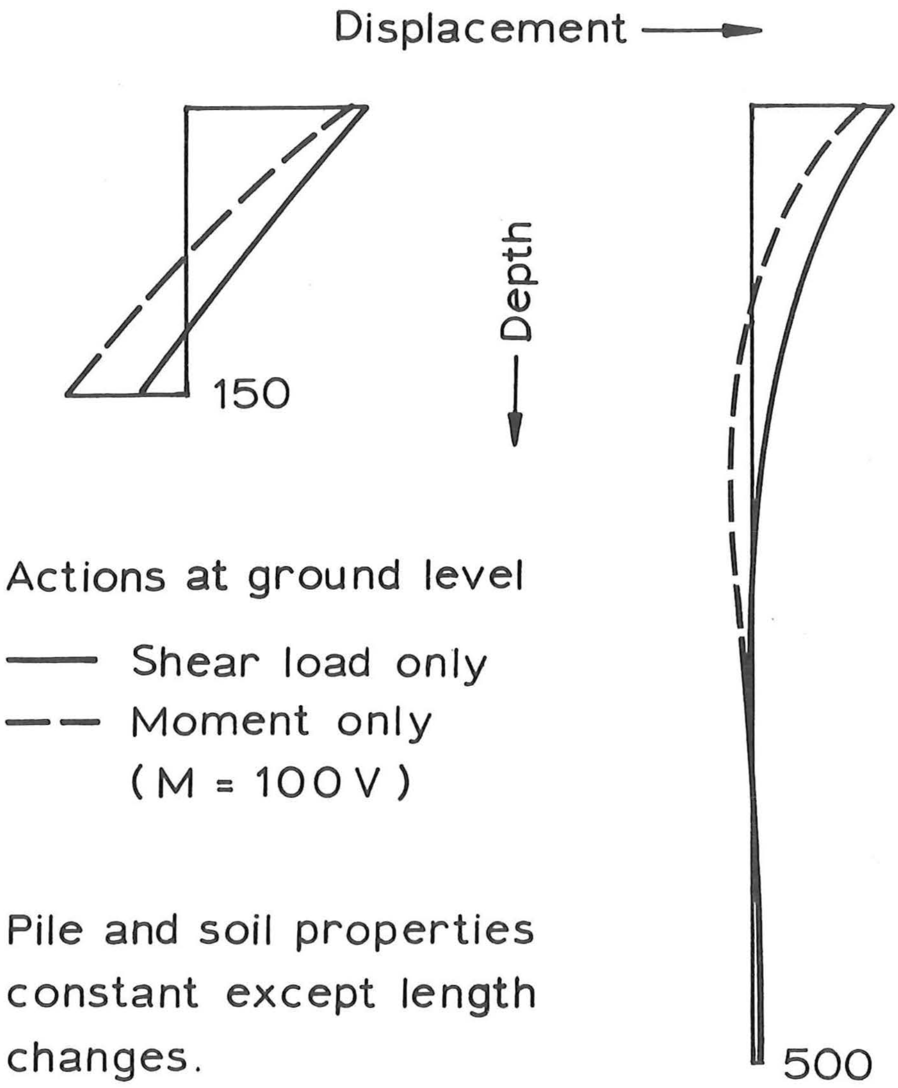


FIG. 3.7 - SOME DISPLACED SHAPES

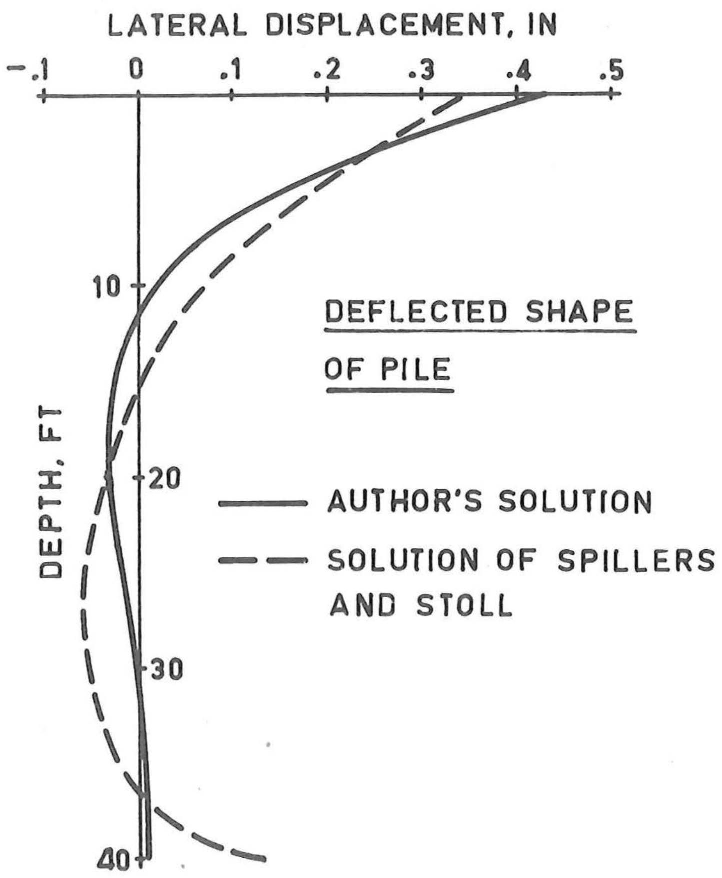


FIG. 3.8 - COMPARISON WITH SOLUTION OF SPILLERS AND STOLL

3.5.5 Comparison with solutions of Poulos

Poulos[7] has recently developed solutions for laterally loaded piles which also use the elastic half-space as a soil model, his method of solution differing from the writer's in several details.

Firstly, he has used a finite difference form of the basic beam equation $\frac{d^4y}{dx^4} = -\frac{w}{EI}$ to describe pile action where the writer has used the integral equation directly. It is felt that a finite difference approach to this problem, although it can deal with a non-uniform pile, cannot be superior in accuracy or simplicity to a direct approach. Secondly, he has used an integration of the appropriate Mindlin equation over a rectangle, given by Douglas and Davis[42], to evaluate the displacement of the rectangle due to a uniform loading on it. In comparison, the solution developed here uses, in effect, a numerical integration technique which can be carried out to any desired accuracy. A study of the effect of mn has already been given: it has shown that the number of point forces considered does not have to be very large for the solution to approach closely from the conservative side the limiting solution for an infinite number of forces.

A third related difference between the two methods concerns the treatment of the pile width. Poulos has used a uniform distribution of soil reaction across the pile, and has stated that this will be reasonable for a flexible pile. Perhaps the statement was made assuming the pile to be flexible across its width, but a pile may be flexible (along its length) but still relatively narrow, and completely stiff across its width. The force distributions across stiff and flexible wide piles were obtained and showed that although the averages of the ratio of edge force to centre force were very similar, the non-linearity was more marked for the flexible pile, as this ratio varied considerably down the pile. In any case, it must be concluded that to allow the reaction to vary across the width will produce a more accurate solution.

Poulos has given typical solutions for stiff and flexible free and fixed head piles and fig. 3.9 compares these solutions with those of the writer. His notation has been used,

$$I_{\rho h} = \frac{\Delta . E_{soil} . L}{V} \quad \text{for a free-head pile}$$

and $$I_{\rho F} = \frac{\Delta . E_{soil} . L}{V} \quad \text{for a fixed-head pile.}$$

The general form of each displaced shape is similar, with the writer's solutions giving higher top displacements and slopes, which at first sight suggests a more flexible soil representation. But the deflected shapes are those of more flexible piles, suggesting a stiffer soil. This apparently contradictory effect may be due to the finite

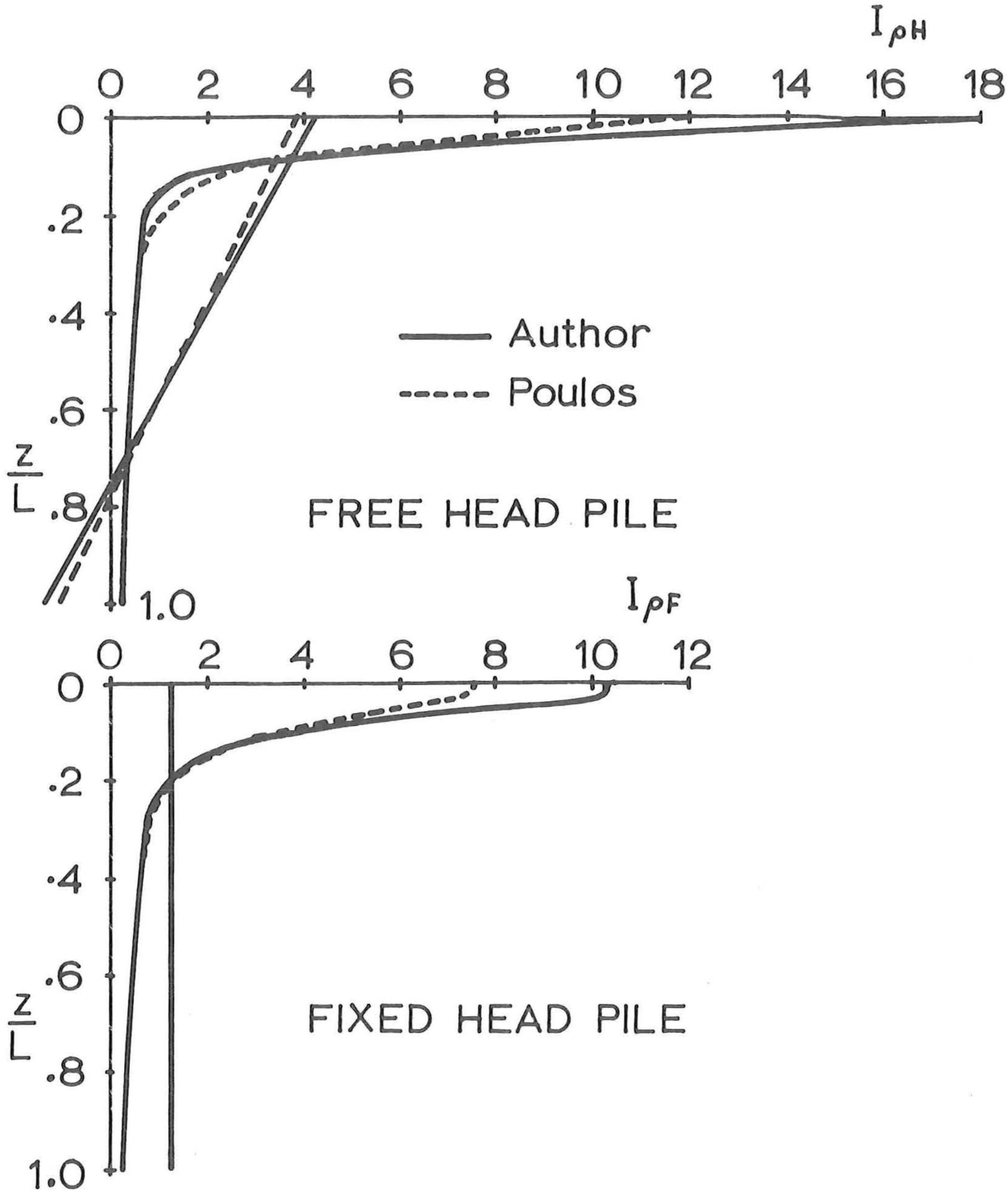


FIG. 3.9 - COMPARISON WITH POULOS' FIG. 8.

difference solution underestimating the top end displacement, or it may be due to the different treatments of width effects.

Although the analysis presented here should be more accurate for an elastic half-space, either may be superior for a real soil, as the force distribution across the pile will be different for cohesive and cohesionless soils. It is possible that the writer's solution, with a non-uniform distribution, would be more accurate for clay, and Poulos', with a uniform distribution, more accurate for sand.

Fig. 3.10 compares both theoretical solutions with two reported tests. The comparisons have been made by applying the same load to the same pile and selecting soil properties so that theoretical and measured top displacements are equal. Pile behaviour as indicated by displacement profiles can then be compared. The tests are those reported by Kerisel and Adam[43] (pile A) and Gleser[20].

It appears from fig. 3.10 that the writer's solution may indeed be more accurate for clay, but it is not easy to say which is better for sand. The soil stiffness appears to vary with depth, and different variations may make either theoretical shape a better fit. In both cases the writer's guessed soil stiffness was higher than Poulos'. The Kerisel and Adam test is discussed again in Chapter 6, when varying soil properties are chosen to give

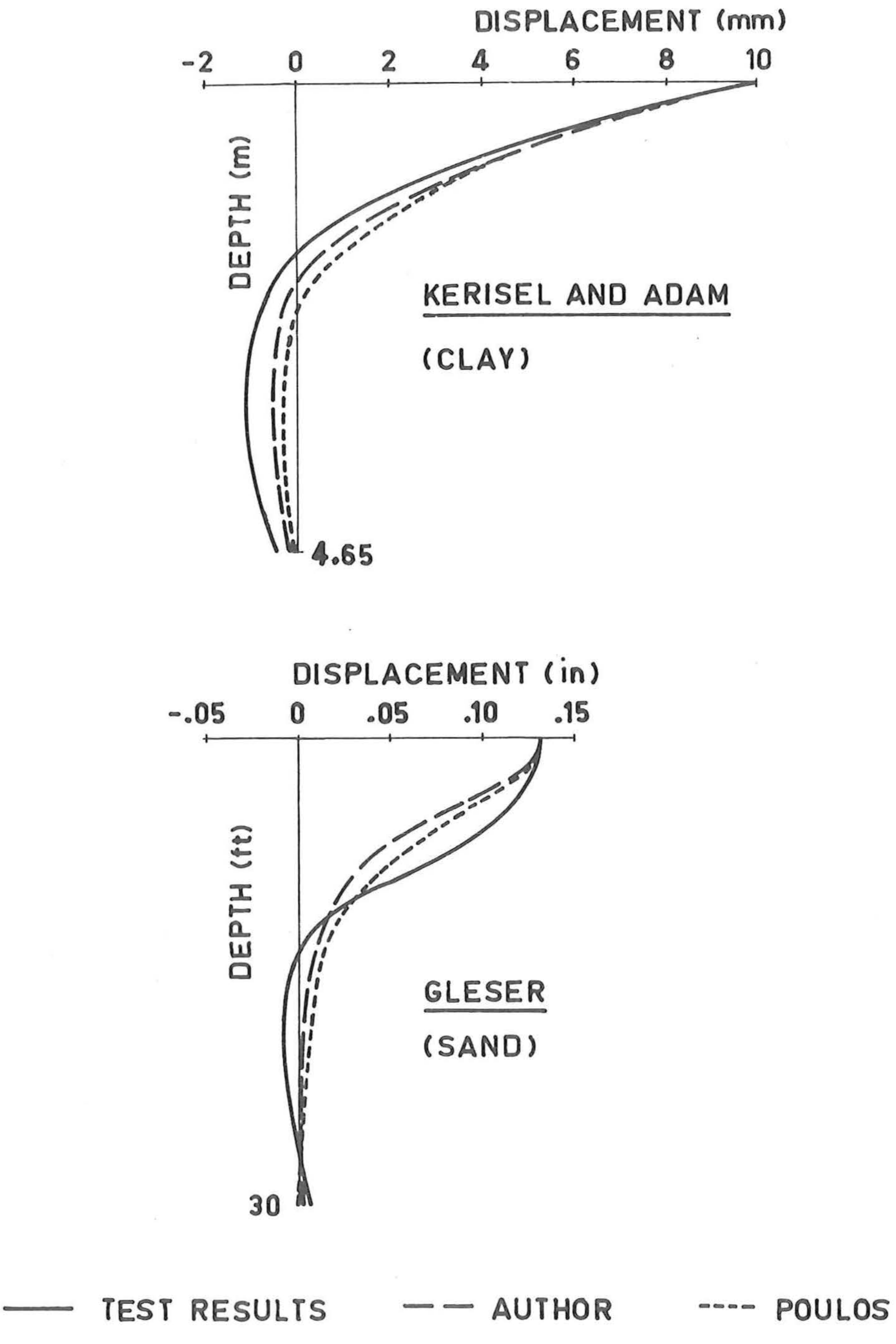


FIG. 3.10 - COMPARISONS OF THEORETICAL SOLUTIONS WITH SOME MEASURED DISPLACEMENT PROFILES.

a proper fit between theory and experiment.

Kerisel and Adam also reported a test with a stiff pile (pile J). It is probably this pile for which Poulos reports a theoretical top rotation 35% less than the measured value. It is not clear on what basis Poulos made his comparison but the writer has made two comparisons. If the same soil properties are used for pile J as for pile A, the theoretical values of both top displacement and rotation are about 90% high, this indicating the soil around pile J to be considerably stiffer than that around pile A. If the comparison is made on the same basis as above, that is, with the soil properties chosen to give equal theoretical and measured top displacements, then the agreement between theoretical and measured top slopes is extremely good, there being less than 3% difference.

The following may be said in summary of this section: Poulos has used an elastic half-space for a soil model, but has employed different techniques to obtain solutions which are similar to those presented here but less conservative in all cases. It is not possible to say which solution fits best with real soil properties but the writer's should be more accurate for an elastic half-space.

CHAPTER FOUR

THE BEHAVIOUR OF A LATERALLY LOADED PILE IN A UNIFORM ELASTIC HALF SPACE

4.1 INTRODUCTION

This chapter examines results obtained using the analysis developed in Chapter 3. The effects of varying pile and soil properties one at a time are examined, and equations relating the ground level displacement and slope to these properties are developed. The equations enable tables to be compiled, and these are included.

4.2 THE EFFECTS PRODUCED ON THE TOP DISPLACEMENT BY VARYING PILE AND SOIL PROPERTIES

4.2.1 Introduction

Fig. 4.1 gives a qualitative idea of how various pile and soil properties will affect the top displacement Δ of a typical pile. The pile is a free-head one, loaded at ground level, so the term top displacement means ground level displacement. The length is fixed, and the pile will be typical at least over part of the ranges considered. The numbers given on the figure are only to provide an idea of the range of values used, as the intention at this stage is simply to obtain some understanding of system behaviour. In particular, it should be noted that origins are not always at zero.

4.2.2 Variation of Pile Width

Fig. 4.1(a) shows that increasing the pile width has the expected effect of decreasing the top displacement. As the width is increased, the soil is effectively stiffened.

4.2.3 Variation of Pile Stiffness

The noteworthy fact to be obtained from fig. 4.1(b) is that the curve is asymptotic to a lowest top displacement, showing that above a certain stiffness the pile may be considered as infinitely stiff. This confirms the existence of rigid pile or pole behaviour.

4.2.4 Variation of Soil Stiffness

Figure 4.1(c) confirms that an increase in soil stiffness, as well as an increase in pile stiffness, will produce a decrease in top deflection. The effects on the displaced shape, though, will obviously be opposite, it depending on the ratio of pile stiffness to soil stiffness. The shear modulus G and Poisson's ratio μ rather than Young's modulus E and μ have been chosen to represent the soil stiffness. There appear to be no strong reasons for the use of either representation (as a conversion is readily made for an elastic soil) although G is perhaps a more natural quantity than E for soil. The result of this choice is to make μ a more significant variable. That is, fig. 4.1(d) has been prepared with G kept constant:

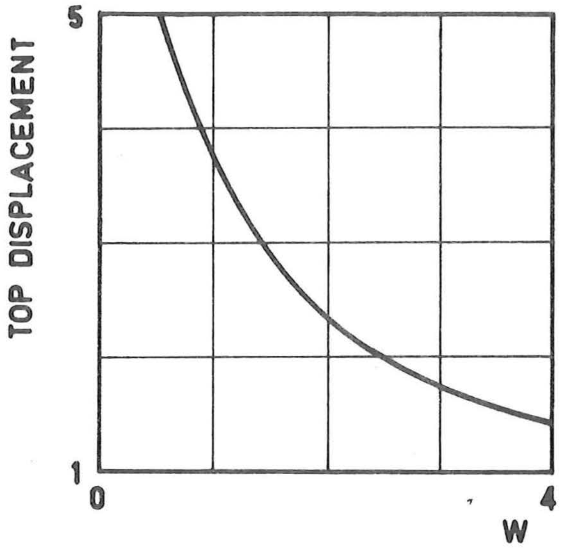


FIG. 4.1a. PILE WIDTH VARIES

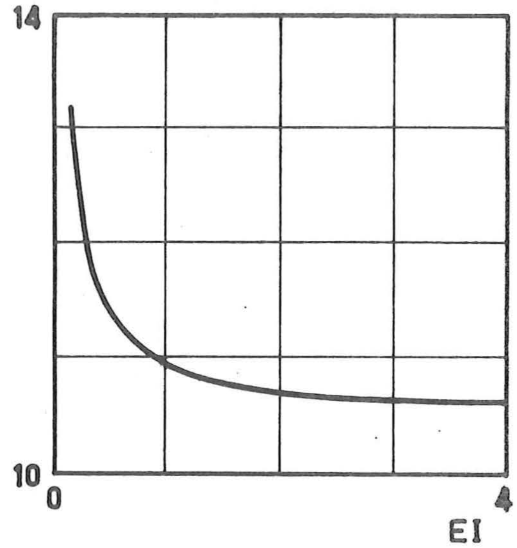


FIG. 4.1b. PILE STIFFNESS VARIES

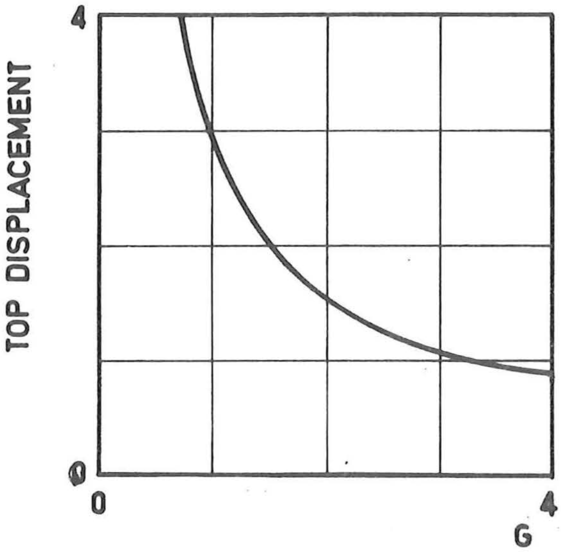


FIG. 4.1c. SHEAR MODULUS OF SOIL VARIES

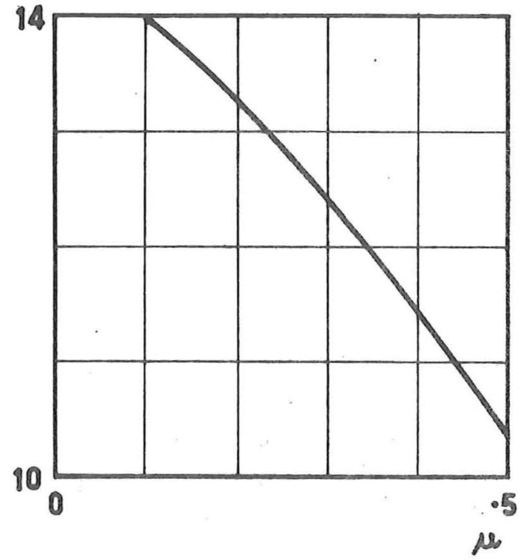


FIG. 4.1d. POISSON'S RATIO OF SOIL VARIES

FIG. 4.1. THE EFFECTS OF PILE AND SOIL PROPERTIES ON THE TOP DISPLACEMENT

if E were kept constant, the significance of μ would be reduced, as has been pointed out by Douglas and Davis[42]. If desired, both figs. 4.1(c) and 4.1(d) may be regarded as plots against E, the first with μ constant and the second with G constant. It is interesting that $\mu = 0.5$ does not seem to be a boundary on the curve in fig. 4.1(d).

4.2.5 Variation of Pile Length

The manner in which pile length influences system behaviour is considered last, as understanding this influence is probably the easiest way to understand the system. A study of the top displacement - pile length curve is later used to develop approximate general rules of system behaviour.

Fig. 4.2 gives two Δ-L plots. Once again the pile used cannot really be called typical because the length varies, but it will be typical over a certain range of length, and the behaviour shown is representative of all pile - soil systems. Numbers are given so that the two plots may be related to one another.

The natural plot shows clearly that above a certain pile length, which may be termed a critical length, there is no change in pile action with an increase in length. This then is a second critical length, the first having been shown from the Δ-EI graph as that below which the pile is rigid. That both these lengths exist has generally been recognized previously.

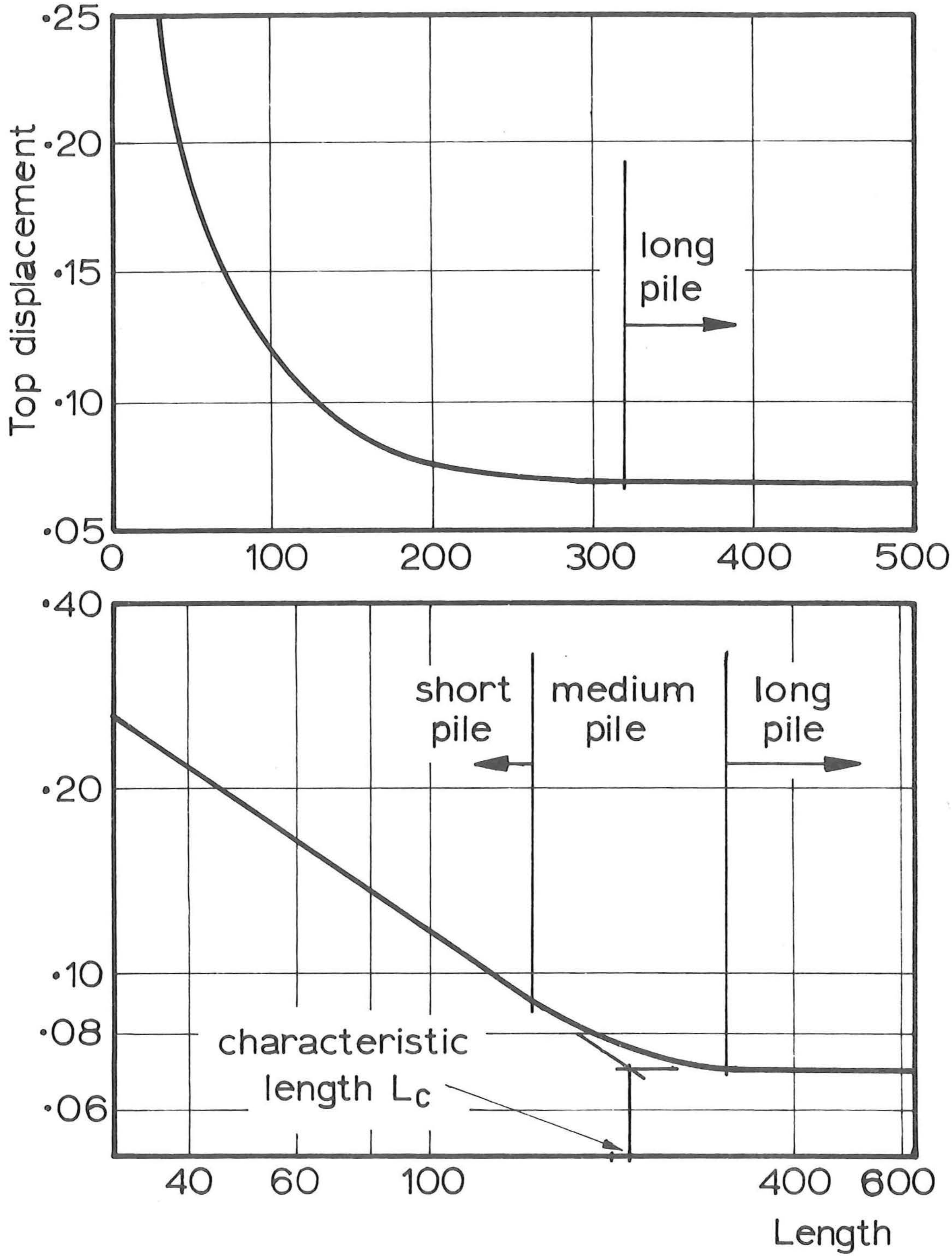


FIG. 4.2.- NATURAL AND LOG-LOG PLOTS OF TOP DISPLACEMENT VS. LENGTH

When the top displacement is plotted against the length on a log-log scale, both critical lengths show up clearly as the transition points between the three distinct parts of the curve, and this plot allows any pile to be readily classed as short (or stiff), medium, or long. This graph further provides an easy definition of a pile characteristic length L_c: it is the length where the two straight portions of the curve intersect.

As the displacement values for fig. 4.2 were obtained with only a shear load at the top of the pile, the definition must also include this fact. Graphs of top displacement Δ_M due to moment loading and top slopes due to shear (θ_V) and moment (θ_M) loading are of similar shape to the one considered, but would all give a lower characteristic length if used for the definition.

As L_c proves to be not only useful as a characteristic length but also significant physically it is defined formally. The characteristic length L_c of a given pile in a given soil is obtained from a log-log plot against pile length of the ground level displacement Δ_V due to a ground level shear load. It is the length at which the two straight portions of the curve intersect.

4.3 EQUATIONS RELATING EFFECTS TO PROPERTIES

Fig. 4.3 shows how pile and soil properties affect the shape of the log Δ_V - log L curve, now considered as bilinear but requiring some correction in the transition

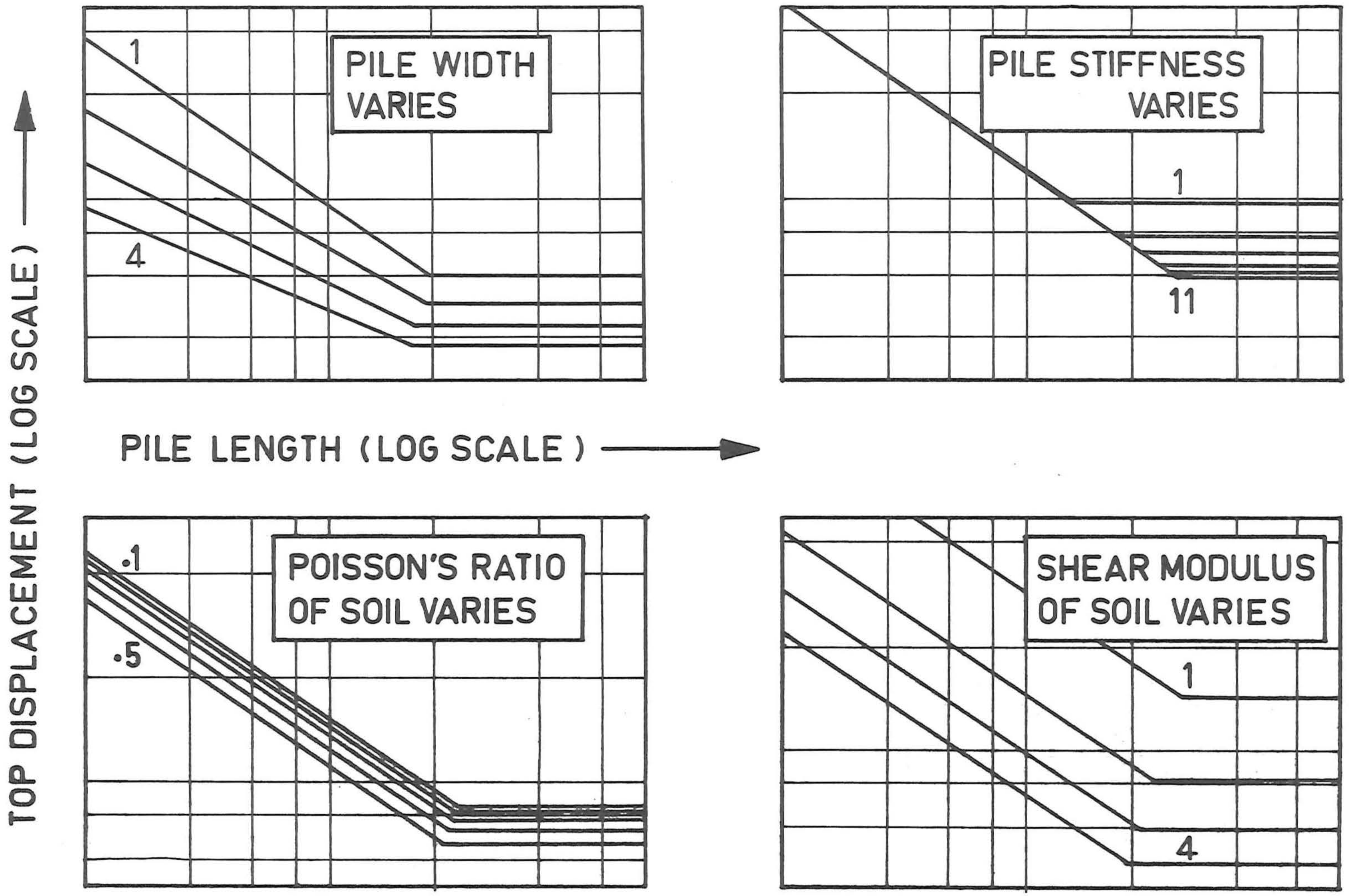


FIG.4.3. THE EFFECT OF PILE AND SOIL PROPERTIES ON THE Δ-L CURVE

zone near $L = L_c$. The member curves of each family are for equal increments of the variable for that family. The regularity of the curves lends encouragement to the idea that it should be possible to formulate equations relating quantities such as Δ_v to pile and soil properties and to the loading. The development of such equations did prove possible, as shown below, and has led to a set of tables which provide the approximate top displacement and slope for any free or fixed-head pile.

The development of each equation was basically a trial and error process and is illustrated by the formulation of an equation for L_c.

L_c is obviously (for a linearly elastic soil) independent of pile loading and can be expected to vary with the pile stiffness EI, the soil properties G and μ, and the pile width W. The five physical quantities in two dimensions indicate that three dimensionless numbers can be found and L_c/W, μ, and EI/GW^4 were the numbers used. By varying μ alone it was found that

$$\log \frac{L_c}{W} = C_1 + .083 \log (.64 - \mu).$$

Similarly it was found that

$$\log \frac{L_c}{W} = C_2 + .28 \log \frac{EI}{GW^4}$$

The two equations were combined to give

$$\log \frac{L_c}{W} = .278 + .28 \log \frac{EI}{GW^4} + .083 \log (.64 - \mu)$$

or $$L_c = 1.9W \left(\frac{EI}{GW^4}\right)^{.28} (.64 - \mu)^{.083} \tag{4.1}$$

Other equations, derived in a similar manner, follow.

$$\log \Delta_v = -2.2 + .48 \log \left(\frac{L_c}{W} + 15\right) + 1.4\,(.72 - \mu)^{.087}$$

$$+ \log \frac{V}{GW} - \left(.24 \log \frac{L_c}{W} + .37\right) \log \frac{L}{W}, \quad L \not> L_c \tag{4.2}$$

$$\theta_v = \Delta_M = \frac{.58\,\Delta_v\,(Lc/L + 1.4)^{1.28}}{.75W + L_c}, \quad L \not> .90L_c \tag{4.3}$$

$$\theta_M = \frac{.88\,\Delta_M\,(Lc/L + 1.4)^{1.28}}{.75W + L_c}, \quad L \not> .51L_c \tag{4.4}$$

Equation (4.2) was derived assuming the Δ_v -L curve to be bilinear and so requires correction when L is close to L_c . This correction c was found to be nearly symmetrical about $L = L_c$, and negligible when $.7L_c < L < 1.3L_c$. The equation derived was

$$c = +\,(2.4\,L_c/W + 36)(.3 - |L-L_c|\,)^2, \quad .7L_c < L < 1.3L_c \tag{4.5}$$

Equations (4.3) and (4.4) were derived using actual values of Δ_v, so they use the corrected value of Δ_v but themselves require no correction.

These five equations were used to form tables so that

$$L_c = W.\,T_1 \tag{4.1a}$$

$$\Delta_v = \frac{V}{GW}\,.T_2 + c, \tag{4.2a}$$

$$\Theta_V = \Delta_M = \frac{.58\,\Delta_V}{W}\,.T_3\ ,\ L \not> .90L_c \tag{4.3a}$$

and $$\Theta_M = \frac{.88\,\Delta_M}{W}\,.T_3\ ,\ L \not> .51L_c\ . \tag{4.4a}$$

Tables for the top displacement Δ_F and the fixing moment M_F for a fixed head pile were written using these equations.

Since if $.51L_c < L < .90L_c$, distinction must be made between T_3 in (4.3a) and T_3 in (4.4a), equation (4.4a) is rewritten as

$$\Theta_M = \frac{.88\,\Delta_M}{W}\ .\ T_4$$

It can then be shown that

$$M_F = -\frac{VW}{.88T_4} \tag{4.5}$$

and $$\Delta_F = \frac{VT_2}{GW}\left(1 - .66\,\frac{T_3}{T_4}\right)\ . \tag{4.6}$$

It is necessary to emphasise that these equations are only approximate. They were derived using a quite small number of significant figures, from a limited amount of data, and including in all equations after the first a dependence on previous approximate equations. That is, each equation is itself accurate to 3 or 4%, but all equations after the first rely on previous ones for their input data and thus the possibility of a cumulative error arises. The error has been found in one instance to be equal to +15% but this is far from normal. The tables can be used with confidence as long as a rough guide only is required: in general they will give better than this. They will certainly be useful as a guide to pile sizes if final design is to be carried out using the computer. That the limitations of the elastic continuum as a soil model should be borne in mind has already been mentioned.

The tables are presented in section 4.5 of this chapter.

4.4 THE SIGNIFICANCE OF THE PILE CHARACTERISTIC LENGTH

The characteristic length of the pile was conceived of as just that: a length characteristic of the pile which enabled the actual length to be expressed as a non-dimensional fraction. It proves, however, to be rather more significant than that.

It is obvious that there will, economically speaking, be an optimum pile for any situation and the analysis which follows shows that the most efficient length for a laterally loaded pile is the characteristic length (although the analysis is simple and must ignore such factors as availability).

It will be remembered that the definition of L_c included the fact that loading should consist of shear alone. If moment loading were used to define a different characteristic length L_{cM} then the relationship

$$L_{cM} = .8L_c$$

would always hold. We will therefore consider several different sized piles under loading V and allow a top displacement Δ. It will be assumed first that the bilinear curve describes system action adequately, and a check can be later made as to whether the correction term affects the conclusion reached.

It is immediately obvious, because the displacement first reaches a minimum at $L = L_c$, that for economy

$$L \ngtr L_c \tag{4.7}$$

It will now be assumed that pile cost is proportional to pile volume. This assumption will be reasonably accurate as regards material cost, and it will also be reasonable as regards driving cost, which will increase with both length and sectional area. It will also be assumed that all piles have the same shape.

Thus $$\text{cost} = k_1 L W^2$$

and the moment of inertia

$$I = k_2 W^4 .$$

If all piles are of the same material,

$$\frac{E}{G} = k_3 .$$

Thus $$\frac{EI}{GW^4} = k_2 k_3$$

and equation (4.1) can be written

$$L_c = k_4 W, \tag{4.8}$$

giving cost $= \dfrac{k_1}{k_4^{\,2}} LL_c^{\,2}.$ (4.9)

Equation (4.2) may now be written

$$\log \Delta_v = -2.2 + .48 \log (k_4 + 15) + 1.4 (.72-\mu)^{.087}$$

$$+ \log \frac{V}{GW} - (.24 \log k_4 + .37) \log \frac{L}{W} ,$$

so $$\Delta_v = k_5 \left(\frac{1}{W}\right)\left(\frac{W}{L}\right)^{k_6}$$

$$= \frac{k_5}{W^{1-k_6} L^{k_6}} .$$

Now for practical cases (see tables) $L_c/W = k_4$ varies between approximately 5 and 70, so the range of $k_6 = .24 \log k_4 + .37$ is approximately .50 - .85.

For a unit top displacement,

$$W^{1-k_6} = \frac{k_5}{L^{k_6}} .$$

That is, from (4.8),

$$L_c^{\,1-k_6} = \frac{k_7}{L^{k_6}}$$

or $$L_c = k_7 L^{k_8} ,$$

where $$k_8 = - \frac{k_6}{1-k_6} .$$

Now if $k_6 = .50$, $k_8 = -1$;

and if $k_6 = .85$, $k_8 = -5.7$;

so $k_8 \leqslant -1$

Substituting into equation (4.9),

$$\text{cost} = \frac{k_1}{k_4^{\,2}} L \, (k_7 \, L^{k_8})^2$$

$$= k_9 \, L^{1+2k_8}$$

$$= k_9 \, L^{k_{10}}, \text{ where } k_{10} \leqslant -1.$$

Thus maximum L will give minimum cost. However, it has already been established that (equation 4.7) $L \ngtr L_c$, and so the most economical length is $L = L_c$. The correction terms near $L = L_c$ are found to be too small to affect this conclusion.

The limitations of this analysis should be noted. The reality of the assumption that cost is proportional to volume can be judged on its merits in any situation. The other assumption is that the design is based on an allowable ground level displacement under shear loading. This has been chosen because it gives the longest efficient length, but if moment loading is applied or top slopes are important or the displacements above ground level are critical, then a shorter stiffer pile will be more efficient. It must be once again remembered that the soil is assumed to be elastic: if yielding occurs, L_c and thus the efficient length will become longer. Also, this analysis does not consider ultimate behaviour: if the design is based on strength considerations L_c may not be the most efficient length.

Considering these limitations, it is still obvious that the characteristic length gives a very useful guide to the most efficient pile length for lateral loading. Some of the items mentioned above could perhaps be treated by the use of a "length factor α", and a pile designed for lateral load would then be driven to a depth of αL_c. Thus, for example, where yielding may occur near the surface, α might be taken as 1.2 or 1.3. Used in this way, L_c could become a very useful design parameter.

4.5 TABLES

This section provides directions on the use of the tables following. The pile given as an example by Spillers and Stoll[6] is used as an example to demonstrate their use. It has $E = 30 \times 10^6$ psi, $I = 4010$ in^4, $W = 14$ in, $L = 40$ft, $G = 1187$ psi, $\mu = .45$, and is loaded with 40,000 lb, 12 ft above ground level. All notation has already been given but is repeated here for clarity:

L_c = characteristic length,

Δ_V = ground level displacement due to a ground level shear load,

Δ_M = ground level displacement due to a ground level moment load,

Δ_F = displacement of a pile fixed at ground level,

θ_V = ground level slope due to shear load,

θ_M = slope due to moment load.

M_F = moment required to fix a pile against rotation at ground level.

Table 1, for obtaining L_c

This table comprises five pages, with increasing values of EI/GW^4. PR = Poisson's ratio. The values tabulated are of L_c/W.

For the example pile $EI/GW^4 = 2610$

and $\mu = .45$

so $L_c/W = 15.0$

and $L_c = 210$ inches

Table 2, for obtaining Δ_V

Table 2 comprises five pages, with Poisson's ratio taking a different value on each page. Tabulated values are Δ_V GW/V. The correction factor from table 4 should be used if $.8L_c < L < 1.2L_c$.

For the example pile $L > L_c$ so the column with L/L_c = 1.0 is used. $L > 1.2L_c$ so no correction is necessary.

$$\text{Table Value} = \frac{.100 + .091}{2} = .096 \quad .$$

$$\text{Therefore } \Delta_V = .096 \frac{V}{GW}$$

$$= 5.8 \times 10^{-6} V.$$

Table 3, for Δ_F

The use of this table is the same as for table 2, tabulated values being Δ_F GW/V.

Table 4, correction factors for tables 2 and 3

Tabulated values are percentage corrections to be added.

Table 5, for Δ_M, θ_V and θ_M.

Tabulated values T are used in the following manner.

$$\theta_V = 0.58 \frac{\Delta_V}{W} . T$$

$$\Delta_M = \frac{.58 \Delta_{vM}}{W} . T$$, where Δ_{vM} is obtained from table 2 using the value of M instead of V.

For these two cases, $L/L_c \not> .90$.

$$\theta_M = \frac{.88 \Delta_m}{W} . T \quad , \frac{L}{L_c} \not> .51$$

For the example pile,

$$\theta_V = \frac{.58 \times 5.8 \times 10^{-6} V}{14} \times .206$$

$$= 4.9 \times 10^{-8} V$$

$$\Delta_M = 4.9 \times 10^{-8} M$$

$$\theta_M = \frac{.88 \times 4.9 \times 10^{-8} M}{14} \times .300$$

$$= 9.3 \times 10^{-10} M$$

Table 6, for M_F

Tabulated values are $M_F GW/V$.

The example pile may now be fully solved:

$V = 4 \times 10^4$ and $M = 4.8 \times 10^6$.

$\Delta = \Delta_V + \Delta_M$

$= 5.8 \times 10^{-6} \times 4 \times 10^4 + 4.9 \times 10^{-8} \times 4.8 \times 10^6$

$= .467$ in.

$\theta = \theta_V + \theta_M$

$= 4.9 \times 10^{-8} \times 4 \times 10^4 + 9.3 \times 10^{-10} \times 4.8 \times 10^6$

$= .641 \times 10^{-2}$

Values obtained directly from the computer program are

$\Delta = .438$

and $\theta = .595 \times 10^{-2}$

so the errors obtained are + 7% and + 8% respectively.

The following table summarises this section.

Table number	Free or fixed head	Obtains	Tabulated	Notes
1	both	L_c	L_c/W	
2	free	Δ_V	$\Delta_V \cdot \frac{GW}{V}$	$L \ngtr L_c$
3	fixed	Δ_F	$\Delta_F \cdot \frac{GW}{V}$	$L \ngtr L_c$
4	both	correction for 2 and 3	% correction (add)	$\cdot 8L_c < L < 1\cdot 2L_c$
5	free	θ_V	$\theta_V \cdot \frac{W}{\cdot 58\Delta_V}$	$L \ngtr \cdot 90L_c$
		Δ_M	$\Delta_M \cdot \frac{W}{\cdot 58\Delta_{VM}}$	$L \ngtr \cdot 90L_c$
		θ_M	$\theta_M \cdot \frac{W}{\cdot 88\Delta_M}$	$L \ngtr \cdot 51L_c$
6	fixed	M_F	$M_F \cdot \frac{GW}{V}$	$L \ngtr \cdot 51L_c$

TABLE 1, FOR OBTAINING LC

FOR DIRECTIONS ON USE SEE INTRODUCTION TO TABLES

PP =	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
EI/GW^4										
250	8.5	8.5	8.4	8.3	8.2	8.2	8.0	7.9	7.8	7.6
500	10.4	10.3	10.2	10.1	10.0	9.9	9.8	9.6	9.4	9.2
750	11.6	11.5	11.4	11.3	11.2	11.1	10.9	10.8	10.6	10.3
1000	12.6	12.5	12.4	12.3	12.2	12.0	11.9	11.7	11.5	11.2
1250	13.4	13.3	13.2	13.1	12.9	12.8	12.6	12.4	12.2	11.9
1500	14.1	14.0	13.9	13.8	13.6	13.5	13.3	13.1	12.8	12.5
1750	14.7	14.6	14.5	14.4	14.2	14.1	13.9	13.7	13.4	13.1
2000	15.3	15.2	15.0	14.9	14.8	14.6	14.4	14.2	13.9	13.6
2250	15.8	15.7	15.5	15.4	15.3	15.1	14.9	14.7	14.4	14.0
2500	16.3	16.1	16.0	15.9	15.7	15.5	15.3	15.1	14.8	14.4
2750	16.7	16.6	16.4	16.3	16.1	16.0	15.7	15.5	15.2	14.8
3000	17.1	17.0	16.9	16.7	16.5	16.3	16.1	15.9	15.6	15.2
3250	17.5	17.4	17.2	17.1	16.9	16.7	16.5	16.2	15.9	15.5
3500	17.9	17.7	17.6	17.4	17.3	17.1	16.8	16.6	16.3	15.9
3750	18.2	18.1	17.9	17.8	17.6	17.4	17.2	16.9	16.6	16.2
4000	18.5	18.4	18.3	18.1	17.9	17.7	17.5	17.2	16.9	16.5
4250	18.9	18.7	18.6	18.4	18.2	18.0	17.8	17.5	17.2	16.7
4500	19.2	19.0	18.9	18.7	18.5	18.3	18.1	17.8	17.4	17.0
4750	19.5	19.3	19.2	19.0	18.8	18.6	18.3	18.1	17.7	17.3
5000	19.7	19.6	19.4	19.3	19.1	18.9	18.6	18.3	18.0	17.5
5250	20.0	19.9	19.7	19.5	19.3	19.1	18.9	18.6	18.2	17.8
5500	20.3	20.1	20.0	19.8	19.6	19.4	19.1	18.8	18.5	18.0
5750	20.5	20.4	20.2	20.0	19.8	19.6	19.4	19.1	18.7	18.2
6000	20.8	20.6	20.5	20.3	20.1	19.8	19.6	19.3	18.9	18.4
6250	21.0	20.9	20.7	20.5	20.3	20.1	19.8	19.5	19.1	18.7
6500	21.2	21.1	20.9	20.7	20.5	20.3	20.0	19.7	19.3	18.9
6750	21.5	21.3	21.1	21.0	20.7	20.5	20.2	19.9	19.5	19.1
7000	21.7	21.5	21.4	21.2	21.0	20.7	20.5	20.1	19.7	19.3
7250	21.9	21.7	21.6	21.4	21.2	20.9	20.7	20.3	19.9	19.4
7500	22.1	22.0	21.8	21.6	21.4	21.1	20.9	20.5	20.1	19.6
7750	22.3	22.2	22.0	21.8	21.6	21.3	21.0	20.7	20.3	19.8
8000	22.5	22.4	22.2	22.0	21.8	21.5	21.2	20.9	20.5	20.0
8250	22.7	22.6	22.4	22.2	21.9	21.7	21.4	21.1	20.7	20.2
8500	22.9	22.7	22.6	22.4	22.1	21.9	21.6	21.3	20.9	20.3
8750	23.1	22.9	22.7	22.5	22.3	22.1	21.8	21.4	21.0	20.5
9000	23.3	23.1	22.9	22.7	22.5	22.2	21.9	21.6	21.2	20.7
9250	23.5	23.3	23.1	22.9	22.7	22.4	22.1	21.8	21.4	20.8
9500	23.6	23.5	23.3	23.1	22.8	22.6	22.3	21.9	21.5	21.0
9750	23.8	23.6	23.4	23.2	23.0	22.7	22.4	22.1	21.7	21.1
10000	24.0	23.8	23.6	23.4	23.2	22.9	22.6	22.2	21.8	21.3

FOR DIRECTIONS ON USE SEE INTRODUCTION TO TABLES

PR =	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
EI/GW^4										
10500	24.3	24.1	23.9	23.7	23.5	23.2	22.9	22.6	22.1	21.6
11000	24.6	24.4	24.2	24.0	23.8	23.5	23.2	22.9	22.4	21.9
11500	24.9	24.7	24.5	24.3	24.1	23.8	23.5	23.1	22.7	22.1
12000	25.2	25.0	24.8	24.6	24.4	24.1	23.8	23.4	23.0	22.4
12500	25.5	25.3	25.1	24.9	24.7	24.4	24.1	23.7	23.2	22.6
13000	25.8	25.6	25.4	25.2	24.9	24.6	24.3	23.9	23.5	22.9
13500	26.1	25.9	25.7	25.4	25.2	24.9	24.6	24.2	23.7	23.1
14000	26.3	26.1	25.9	25.7	25.5	25.2	24.8	24.4	24.0	23.4
14500	26.6	26.4	26.2	26.0	25.7	25.4	25.1	24.7	24.2	23.6
15000	26.9	26.7	26.4	26.2	25.9	25.7	25.3	24.9	24.4	23.8
15500	27.1	26.9	26.7	26.5	26.2	25.9	25.6	25.2	24.7	24.1
16000	27.3	27.1	26.9	26.7	26.4	26.1	25.8	25.4	24.9	24.3
16500	27.6	27.4	27.2	26.9	26.7	26.3	26.0	25.6	25.1	24.5
17000	27.8	27.6	27.4	27.1	26.9	26.6	26.2	25.8	25.3	24.7
17500	28.0	27.8	27.6	27.4	27.1	26.8	26.4	26.0	25.5	24.9
18000	28.3	28.1	27.8	27.6	27.3	27.0	26.6	26.2	25.7	25.1
18500	28.5	28.3	28.0	27.8	27.5	27.2	26.8	26.4	25.9	25.3
19000	28.7	28.5	28.3	28.0	27.7	27.4	27.1	26.6	26.1	25.5
19500	28.9	28.7	28.5	28.2	27.9	27.6	27.2	26.8	26.3	25.7
20000	29.1	28.9	28.7	28.4	28.1	27.8	27.4	27.0	26.5	25.8
20500	29.3	29.1	28.9	28.6	28.3	28.0	27.6	27.2	26.7	26.0
21000	29.5	29.3	29.1	28.8	28.5	28.2	27.8	27.4	26.9	26.2
21500	29.7	29.5	29.2	29.0	28.7	28.4	28.0	27.6	27.0	26.4
22000	29.9	29.7	29.4	29.2	28.9	28.6	28.2	27.7	27.2	26.5
22500	30.1	29.9	29.6	29.4	29.1	28.7	28.4	27.9	27.4	26.7
23000	30.3	30.0	29.8	29.5	29.2	28.9	28.5	28.1	27.6	26.9
23500	30.5	30.2	30.0	29.7	29.4	29.1	28.7	28.3	27.7	27.0
24000	30.6	30.4	30.2	29.9	29.6	29.3	28.9	28.4	27.9	27.2
24500	30.8	30.6	30.3	30.1	29.8	29.4	29.0	28.6	28.0	27.3
25000	31.0	30.8	30.5	30.2	29.9	29.6	29.2	28.8	28.2	27.5
25500	31.2	30.9	30.7	30.4	30.1	29.8	29.4	28.9	28.4	27.7
26000	31.3	31.1	30.8	30.6	30.3	29.9	29.5	29.1	28.5	27.8
26500	31.5	31.3	31.0	30.7	30.4	30.1	29.7	29.2	28.7	28.0
27000	31.7	31.4	31.2	30.9	30.6	30.2	29.8	29.4	28.8	28.1
27500	31.8	31.6	31.3	31.1	30.7	30.4	30.0	29.5	29.0	28.2
28000	32.0	31.8	31.5	31.2	30.9	30.6	30.2	29.7	29.1	28.4
28500	32.1	31.9	31.7	31.4	31.1	30.7	30.3	29.8	29.3	28.5
29000	32.3	32.1	31.8	31.5	31.2	30.9	30.5	30.0	29.4	28.7
29500	32.5	32.2	32.0	31.7	31.4	31.0	30.6	30.1	29.5	28.8
30000	32.6	32.4	32.1	31.8	31.5	31.2	30.7	30.3	29.7	28.9

FOR DIRECTIONS ON USE SEE INTRODUCTION TO TABLES

PR =	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
EI/GW4										
31000	32.9	32.7	32.4	32.1	31.8	31.4	31.0	30.5	30.0	29.2
32000	33.2	33.0	32.7	32.4	32.1	31.7	31.3	30.8	30.2	29.5
33000	33.5	33.2	33.0	32.7	32.4	32.0	31.6	31.1	30.5	29.7
34000	33.8	33.5	33.3	33.0	32.6	32.3	31.8	31.3	30.7	30.0
35000	34.0	33.8	33.5	33.2	32.9	32.5	32.1	31.6	31.0	30.2
36000	34.3	34.1	33.8	33.5	33.2	32.8	32.4	31.8	31.2	30.5
37000	34.6	34.3	34.1	33.7	33.4	33.0	32.6	32.1	31.5	30.7
38000	34.8	34.6	34.3	34.0	33.7	33.3	32.8	32.3	31.7	30.9
39000	35.1	34.8	34.6	34.2	33.9	33.5	33.1	32.6	31.9	31.1
40000	35.3	35.1	34.8	34.5	34.1	33.8	33.3	32.8	32.2	31.4
41000	35.6	35.3	35.0	34.7	34.4	34.0	33.6	33.0	32.4	31.6
42000	35.8	35.6	35.3	35.0	34.6	34.2	33.8	33.3	32.6	31.8
43000	36.1	35.8	35.5	35.2	34.8	34.5	34.0	33.5	32.8	32.0
44000	36.3	36.0	35.7	35.4	35.1	34.7	34.2	33.7	33.0	32.2
45000	36.5	36.3	36.0	35.6	35.3	34.9	34.4	33.9	33.2	32.4
46000	36.8	36.5	36.2	35.9	35.5	35.1	34.7	34.1	33.5	32.6
47000	37.0	36.7	36.4	36.1	35.7	35.3	34.9	34.3	33.7	32.8
48000	37.2	36.9	36.6	36.3	35.9	35.5	35.1	34.5	33.9	33.0
49000	37.4	37.1	36.8	36.5	36.1	35.7	35.3	34.7	34.1	33.2
50000	37.6	37.3	37.0	36.7	36.4	35.9	35.5	34.9	34.2	33.4
51000	37.8	37.6	37.3	36.9	36.6	36.1	35.7	35.1	34.4	33.6
52000	38.0	37.8	37.5	37.1	36.8	36.3	35.9	35.3	34.6	33.8
53000	38.2	38.0	37.7	37.3	36.9	36.5	36.1	35.5	34.8	33.9
54000	38.4	38.2	37.9	37.5	37.1	36.7	36.2	35.7	35.0	34.1
55000	38.6	38.4	38.0	37.7	37.3	36.9	36.4	35.9	35.2	34.3
56000	38.8	38.6	38.2	37.9	37.5	37.1	36.6	36.0	35.3	34.5
57000	39.0	38.7	38.4	38.1	37.7	37.3	36.8	36.2	35.5	34.6
58000	39.2	38.9	38.6	38.3	37.9	37.5	37.0	36.4	35.7	34.8
59000	39.4	39.1	38.8	38.5	38.1	37.6	37.2	36.6	35.9	35.0
60000	39.6	39.3	39.0	38.6	38.3	37.8	37.3	36.7	36.0	35.1
61000	39.8	39.5	39.2	38.8	38.4	38.0	37.5	36.9	36.2	35.3
62000	40.0	39.7	39.3	39.0	38.6	38.2	37.7	37.1	36.4	35.5
63000	40.1	39.8	39.5	39.2	38.8	38.3	37.8	37.2	36.5	35.6
64000	40.3	40.0	39.7	39.3	39.0	38.5	38.0	37.4	36.7	35.8
65000	40.5	40.2	39.9	39.5	39.1	38.7	38.2	37.6	36.9	35.9
66000	40.7	40.4	40.0	39.7	39.3	38.8	38.3	37.7	37.0	36.1
67000	40.8	40.5	40.2	39.9	39.5	39.0	38.5	37.9	37.2	36.2
68000	41.0	40.7	40.4	40.0	39.6	39.2	38.7	38.1	37.3	36.4
69000	41.2	40.9	40.5	40.2	39.8	39.3	38.8	38.2	37.5	36.5
70000	41.3	41.0	40.7	40.3	39.9	39.5	39.0	38.4	37.6	36.7

TABLE 1, FOR OBTAINING LC

FOR DIRECTIONS ON USE SEE INTRODUCTION TO TABLES

EI/GW^4 \ PR =	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
72000	41.7	41.4	41.0	40.7	40.3	39.8	39.3	38.7	37.9	37.0
74000	42.0	41.7	41.3	41.0	40.6	40.1	39.6	39.0	38.2	37.3
76000	42.3	42.0	41.7	41.3	40.9	40.4	39.9	39.3	38.5	37.5
78000	42.6	42.3	42.0	41.6	41.2	40.7	40.2	39.5	38.8	37.8
80000	42.9	42.6	42.3	41.9	41.5	41.0	40.5	39.8	39.1	38.1
82000	43.2	42.9	42.6	42.2	41.8	41.3	40.7	40.1	39.3	38.3
84000	43.5	43.2	42.8	42.5	42.0	41.6	41.0	40.4	39.6	38.6
86000	43.8	43.5	43.1	42.7	42.3	41.8	41.3	40.6	39.9	38.9
88000	44.1	43.8	43.4	43.0	42.6	42.1	41.6	40.9	40.1	39.1
90000	44.4	44.0	43.7	43.3	42.9	42.4	41.8	41.2	40.4	39.4
92000	44.6	44.3	43.9	43.6	43.1	42.6	42.1	41.4	40.6	39.6
94000	44.9	44.6	44.2	43.8	43.4	42.9	42.3	41.7	40.9	39.8
96000	45.2	44.8	44.5	44.1	43.6	43.1	42.6	41.9	41.1	40.1
98000	45.4	45.1	44.7	44.3	43.9	43.4	42.8	42.2	41.3	40.3
100000	45.7	45.3	45.0	44.6	44.1	43.6	43.1	42.4	41.6	40.5
102000	45.9	45.6	45.2	44.8	44.4	43.9	43.3	42.6	41.8	40.8
104000	46.2	45.8	45.5	45.1	44.6	44.1	43.5	42.9	42.0	41.0
106000	46.4	46.1	45.7	45.3	44.9	44.4	43.8	43.1	42.3	41.2
108000	46.7	46.3	46.0	45.6	45.1	44.6	44.0	43.3	42.5	41.4
110000	46.9	46.6	46.2	45.8	45.3	44.8	44.2	43.5	42.7	41.6
112000	47.2	46.8	46.4	46.0	45.6	45.0	44.5	43.8	42.9	41.8
114000	47.4	47.0	46.7	46.2	45.8	45.3	44.7	44.0	43.1	42.1
116000	47.6	47.3	46.9	46.5	46.0	45.5	44.9	44.2	43.3	42.3
118000	47.8	47.5	47.1	46.7	46.2	45.7	45.1	44.4	43.6	42.5
120000	48.1	47.7	47.3	46.9	46.4	45.9	45.3	44.6	43.8	42.7
122000	48.3	47.9	47.6	47.1	46.7	46.1	45.5	44.8	44.0	42.9
124000	48.5	48.2	47.8	47.3	46.9	46.3	45.7	45.0	44.2	43.1
126000	48.7	48.4	48.0	47.6	47.1	46.6	45.9	45.2	44.4	43.2
128000	48.9	48.6	48.2	47.8	47.3	46.8	46.1	45.4	44.6	43.4
130000	49.2	48.8	48.4	48.0	47.5	47.0	46.3	45.6	44.8	43.6
132000	49.4	49.0	48.6	48.2	47.7	47.2	46.5	45.8	44.9	43.8
134000	49.6	49.2	48.8	48.4	47.9	47.4	46.7	46.0	45.1	44.0
136000	49.8	49.4	49.0	48.6	48.1	47.6	46.9	46.2	45.3	44.2
138000	50.0	49.6	49.2	48.8	48.3	47.8	47.1	46.4	45.5	44.4
140000	50.2	49.8	49.4	49.0	48.5	47.9	47.3	46.6	45.7	44.5
142000	50.4	50.0	49.6	49.2	48.7	48.1	47.5	46.8	45.9	44.7
144000	50.6	50.2	49.8	49.4	48.9	48.3	47.7	47.0	46.1	44.9
146000	50.8	50.4	50.0	49.6	49.1	48.5	47.9	47.1	46.2	45.1
148000	51.0	50.6	50.2	49.8	49.3	48.7	48.1	47.3	46.4	45.2
150000	51.2	50.8	50.4	49.9	49.4	48.9	48.2	47.5	46.6	45.4

FOR DIRECTIONS ON USE SEE INTRODUCTION TO TABLES

PR =	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
EI/GW^4										
155000	51.6	51.3	50.9	50.4	49.9	49.3	48.7	47.9	47.0	45.8
160000	52.1	51.7	51.3	50.9	50.3	49.8	49.1	48.4	47.4	46.2
165000	52.6	52.2	51.8	51.3	50.8	50.2	49.5	48.8	47.8	46.6
170000	53.0	52.6	52.2	51.7	51.2	50.6	50.0	49.2	48.2	47.0
175000	53.4	53.0	52.6	52.1	51.6	51.0	50.4	49.6	48.6	47.4
180000	53.9	53.5	53.0	52.6	52.0	51.4	50.8	50.0	49.0	47.8
185000	54.3	53.9	53.4	53.0	52.4	51.8	51.2	50.4	49.4	48.2
190000	54.7	54.3	53.8	53.4	52.8	52.2	51.5	50.7	49.8	48.5
195000	55.1	54.7	54.2	53.7	53.2	52.6	51.9	51.1	50.1	48.9
200000	55.5	55.1	54.6	54.1	53.6	53.0	52.3	51.5	50.5	49.2
205000	55.9	55.4	55.0	54.5	54.0	53.4	52.7	51.8	50.8	49.6
210000	56.2	55.8	55.4	54.9	54.3	53.7	53.0	52.2	51.2	49.9
215000	56.6	56.2	55.7	55.2	54.7	54.1	53.4	52.5	51.5	50.2
220000	57.0	56.5	56.1	55.6	55.0	54.4	53.7	52.9	51.9	50.6
225000	57.3	56.9	56.4	55.9	55.4	54.8	54.0	53.2	52.2	50.9
230000	57.7	57.3	56.8	56.3	55.7	55.1	54.4	53.5	52.5	51.2
235000	58.0	57.6	57.1	56.6	56.1	55.4	54.7	53.9	52.8	51.5
240000	58.4	57.9	57.5	57.0	56.4	55.8	55.0	54.2	53.1	51.8
245000	58.7	58.3	57.8	57.3	56.7	56.1	55.3	54.5	53.4	52.1
250000	59.0	58.6	58.1	57.6	57.0	56.4	55.7	54.8	53.7	52.4
255000	59.4	58.9	58.5	57.9	57.4	56.7	56.0	55.1	54.0	52.7
260000	59.7	59.3	58.8	58.3	57.7	57.0	56.3	55.4	54.3	53.0
265000	60.0	59.6	59.1	58.6	58.0	57.3	56.6	55.7	54.6	53.3
270000	60.3	59.9	59.4	58.9	58.3	57.6	56.9	56.0	54.9	53.5
275000	60.6	60.2	59.7	59.2	58.6	57.9	57.2	56.3	55.2	53.8
280000	60.9	60.5	60.0	59.5	58.9	58.2	57.5	56.6	55.5	54.1
285000	61.2	60.8	60.3	59.8	59.2	58.5	57.7	56.8	55.8	54.4
290000	61.5	61.1	60.6	60.1	59.5	58.8	58.0	57.1	56.0	54.6
295000	61.8	61.4	60.9	60.4	59.8	59.1	58.3	57.4	56.3	54.9
300000	62.1	61.7	61.2	60.6	60.0	59.4	58.6	57.7	56.6	55.1
305000	62.4	62.0	61.5	60.9	60.3	59.6	58.8	57.9	56.8	55.4
310000	62.7	62.2	61.7	61.2	60.6	59.9	59.1	58.2	57.1	55.6
315000	63.0	62.5	62.0	61.5	60.9	60.2	59.4	58.5	57.3	55.9
320000	63.3	62.8	62.3	61.7	61.1	60.4	59.6	58.7	57.6	56.1
325000	63.5	63.1	62.6	62.0	61.4	60.7	59.9	59.0	57.8	56.4
330000	63.8	63.3	62.8	62.3	61.7	61.0	60.2	59.2	58.1	56.6
335000	64.1	63.6	63.1	62.5	61.9	61.2	60.4	59.5	58.3	56.9
340000	64.3	63.9	63.4	62.8	62.2	61.5	60.7	59.7	58.6	57.1
345000	64.6	64.1	63.6	63.1	62.4	61.7	60.9	60.0	58.8	57.3
350000	64.9	64.4	63.9	63.3	62.7	62.0	61.2	60.2	59.1	57.6

TABLE 2, FOR TOP DISPLACEMENT UNDER SHEAR LOADING, FREE HEAD PILE

PAGE 1, POISSON'S RATIO = .10

LC/W \ L/LC =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
6	.778	.529	.422	.360	.318	.287	.263	.245	.229	.216
7	.735	.494	.392	.332	.292	.263	.241	.223	.209	.196
8	.697	.464	.366	.309	.271	.244	.223	.206	.192	.181
9	.665	.439	.344	.290	.254	.227	.207	.191	.178	.167
10	.637	.417	.326	.273	.239	.213	.194	.179	.167	.156
11	.612	.398	.310	.259	.226	.201	.183	.169	.157	.147
12	.589	.381	.295	.246	.214	.191	.173	.159	.148	.138
13	.569	.366	.282	.235	.204	.182	.165	.151	.140	.131
14	.550	.352	.271	.225	.195	.173	.157	.144	.133	.125
15	.533	.339	.261	.216	.187	.166	.150	.137	.127	.119
16	.518	.328	.251	.208	.179	.159	.144	.132	.122	.114
17	.504	.318	.242	.200	.173	.153	.138	.126	.117	.109
18	.490	.308	.235	.193	.166	.147	.133	.121	.112	.105
19	.478	.299	.227	.187	.161	.142	.128	.117	.108	.101
20	.466	.291	.220	.181	.156	.137	.124	.113	.104	.097
21	.456	.283	.214	.176	.151	.133	.120	.109	.101	.094
22	.445	.276	.208	.171	.146	.129	.116	.106	.097	.090
23	.436	.269	.203	.166	.142	.125	.112	.102	.094	.088
24	.427	.262	.197	.161	.138	.121	.109	.099	.091	.085
25	.418	.256	.193	.157	.134	.118	.106	.096	.089	.082
26	.410	.251	.188	.153	.131	.115	.103	.094	.086	.080
27	.402	.245	.184	.150	.128	.112	.100	.091	.084	.078
28	.395	.240	.179	.146	.124	.109	.098	.089	.082	.076
29	.388	.235	.176	.143	.121	.107	.095	.087	.080	.074
30	.381	.231	.172	.140	.119	.104	.093	.084	.078	.072
31	.374	.226	.168	.137	.116	.102	.091	.082	.076	.070
32	.368	.222	.165	.134	.114	.099	.089	.081	.074	.068
33	.362	.218	.162	.131	.111	.097	.087	.079	.072	.067
34	.357	.214	.159	.128	.109	.095	.085	.077	.071	.065
35	.351	.210	.156	.126	.107	.093	.083	.075	.069	.064
36	.346	.207	.153	.123	.105	.091	.081	.074	.068	.062
37	.341	.203	.150	.121	.103	.089	.080	.072	.066	.061
38	.336	.200	.148	.119	.101	.088	.078	.071	.065	.060
39	.331	.197	.145	.117	.099	.086	.077	.069	.063	.059
40	.327	.194	.143	.115	.097	.085	.075	.068	.062	.057
42	.318	.188	.138	.111	.094	.082	.073	.066	.060	.055
44	.310	.182	.134	.107	.091	.079	.070	.063	.058	.053
46	.302	.177	.130	.104	.088	.076	.068	.061	.056	.051
48	.295	.173	.126	.101	.085	.074	.065	.059	.054	.050
50	.288	.168	.123	.098	.082	.072	.063	.057	.052	.048
52	.282	.164	.119	.095	.080	.069	.062	.055	.051	.047
54	.276	.160	.116	.093	.078	.067	.060	.054	.049	.045
56	.270	.156	.113	.090	.076	.066	.058	.052	.048	.044
58	.264	.152	.110	.088	.074	.064	.056	.051	.046	.043
60	.259	.149	.108	.086	.072	.062	.055	.049	.045	.041

TABLE 2, FOR TOP DISPLACEMENT UNDER SHEAR LOADING, FREE HEAD PILE

PAGE 2, POISSON'S RATIO = .20

LC/W	L/LC = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
6	.743	.505	.403	.343	.303	.274	.251	.233	.219	.206
7	.701	.471	.374	.317	.279	.251	.230	.213	.199	.187
8	.665	.443	.349	.295	.259	.233	.212	.196	.183	.172
9	.635	.419	.329	.277	.242	.217	.198	.183	.170	.160
10	.609	.398	.311	.261	.228	.204	.185	.171	.159	.149
11	.584	.380	.295	.247	.215	.192	.175	.161	.149	.140
12	.562	.363	.282	.235	.204	.182	.165	.152	.141	.132
13	.543	.349	.269	.224	.195	.173	.157	.144	.134	.125
14	.525	.336	.259	.215	.186	.165	.150	.137	.127	.119
15	.509	.324	.249	.206	.178	.158	.143	.131	.121	.113
16	.494	.313	.240	.198	.171	.152	.137	.126	.116	.108
17	.481	.303	.231	.191	.165	.146	.132	.120	.111	.104
18	.468	.294	.224	.184	.159	.141	.127	.116	.107	.100
19	.456	.285	.217	.178	.153	.136	.122	.112	.103	.096
20	.445	.277	.210	.173	.148	.131	.118	.108	.099	.092
21	.435	.270	.204	.168	.144	.127	.114	.104	.096	.089
22	.425	.263	.199	.163	.139	.123	.110	.101	.093	.086
23	.416	.256	.193	.158	.135	.119	.107	.098	.090	.084
24	.407	.250	.188	.154	.132	.116	.104	.095	.087	.081
25	.399	.245	.184	.150	.128	.113	.101	.092	.085	.079
26	.391	.239	.179	.146	.125	.110	.098	.089	.082	.076
27	.384	.234	.175	.143	.122	.107	.096	.087	.080	.074
28	.377	.229	.171	.139	.119	.104	.093	.085	.078	.072
29	.370	.224	.168	.136	.116	.102	.091	.083	.076	.070
30	.363	.220	.164	.133	.113	.099	.089	.081	.074	.069
31	.357	.216	.161	.130	.111	.097	.087	.079	.072	.067
32	.351	.212	.157	.128	.108	.095	.085	.077	.070	.065
33	.346	.208	.154	.125	.106	.093	.083	.075	.069	.064
34	.340	.204	.151	.122	.104	.091	.081	.073	.067	.062
35	.335	.201	.149	.120	.102	.089	.079	.072	.066	.061
36	.330	.197	.146	.118	.100	.087	.078	.070	.064	.060
37	.325	.194	.143	.116	.098	.085	.076	.069	.063	.058
38	.321	.191	.141	.113	.096	.084	.075	.068	.062	.057
39	.316	.188	.138	.111	.094	.082	.073	.066	.061	.056
40	.312	.185	.136	.110	.093	.081	.072	.065	.059	.055
42	.303	.179	.132	.106	.089	.078	.069	.063	.057	.053
44	.296	.174	.128	.102	.086	.075	.067	.060	.055	.051
46	.288	.169	.124	.099	.084	.073	.065	.058	.053	.049
48	.281	.165	.120	.096	.081	.070	.062	.056	.051	.047
50	.275	.160	.117	.094	.079	.068	.061	.055	.050	.046
52	.269	.156	.114	.091	.076	.066	.059	.053	.048	.044
54	.263	.152	.111	.088	.074	.064	.057	.051	.047	.043
56	.257	.149	.108	.086	.072	.063	.055	.050	.045	.042
58	.252	.145	.105	.084	.070	.061	.054	.048	.044	.041
60	.247	.142	.103	.082	.069	.059	.052	.047	.043	.039

TABLE 2, FOR TOP DISPLACEMENT UNDER SHEAR LOADING, FREE HEAD PILE

PAGE 3, POISSON'S RATIO = .30

L/LC =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
LC/W										
6	.702	.477	.381	.324	.287	.259	.238	.221	.207	.195
7	.663	.446	.353	.300	.264	.237	.217	.201	.188	.177
8	.629	.419	.330	.279	.245	.220	.201	.186	.173	.163
9	.600	.396	.311	.262	.229	.205	.187	.173	.161	.151
10	.574	.376	.294	.247	.215	.193	.175	.162	.150	.141
11	.552	.359	.279	.234	.203	.182	.165	.152	.141	.132
12	.531	.344	.266	.222	.193	.172	.156	.144	.133	.125
13	.513	.330	.255	.212	.184	.164	.148	.136	.126	.118
14	.496	.317	.244	.203	.176	.156	.142	.130	.120	.112
15	.481	.306	.235	.195	.168	.150	.135	.124	.115	.107
16	.467	.296	.227	.187	.162	.143	.130	.119	.110	.102
17	.454	.286	.219	.181	.156	.138	.124	.114	.105	.098
18	.442	.278	.212	.174	.150	.133	.120	.110	.101	.094
19	.431	.270	.205	.169	.145	.128	.116	.106	.097	.091
20	.421	.262	.199	.163	.140	.124	.112	.102	.094	.087
21	.411	.255	.193	.158	.136	.120	.108	.098	.091	.084
22	.402	.249	.188	.154	.132	.116	.104	.095	.088	.082
23	.393	.242	.183	.150	.128	.113	.101	.092	.085	.079
24	.385	.237	.178	.146	.124	.110	.098	.090	.082	.077
25	.377	.231	.174	.142	.121	.107	.096	.087	.080	.074
26	.370	.226	.170	.138	.118	.104	.093	.085	.078	.072
27	.363	.221	.166	.135	.115	.101	.090	.082	.076	.070
28	.356	.217	.162	.132	.112	.098	.088	.080	.074	.068
29	.350	.212	.158	.129	.110	.096	.086	.078	.072	.066
30	.344	.208	.155	.126	.107	.094	.084	.076	.070	.065
31	.338	.204	.152	.123	.105	.092	.082	.074	.068	.063
32	.332	.200	.149	.121	.102	.090	.080	.073	.067	.062
33	.327	.196	.146	.118	.100	.088	.078	.071	.065	.060
34	.322	.193	.143	.116	.098	.086	.077	.069	.064	.059
35	.317	.190	.140	.113	.096	.084	.075	.068	.062	.058
36	.312	.186	.138	.111	.094	.082	.073	.066	.061	.056
37	.307	.183	.135	.109	.092	.081	.072	.065	.060	.055
38	.303	.180	.133	.107	.091	.079	.071	.064	.058	.054
39	.299	.177	.131	.105	.089	.078	.069	.063	.057	.053
40	.295	.175	.129	.104	.087	.076	.068	.061	.056	.052
42	.287	.169	.125	.100	.084	.074	.065	.059	.054	.050
44	.279	.165	.121	.097	.082	.071	.063	.057	.052	.048
46	.273	.160	.117	.094	.079	.069	.061	.055	.050	.046
48	.266	.156	.114	.091	.077	.067	.059	.053	.049	.045
50	.260	.152	.111	.088	.074	.065	.057	.052	.047	.043
52	.254	.148	.108	.086	.072	.063	.055	.050	.046	.042
54	.249	.144	.105	.084	.070	.061	.054	.049	.044	.041
56	.243	.141	.102	.081	.068	.059	.052	.047	.043	.039
58	.238	.137	.100	.079	.066	.058	.051	.046	.042	.038
60	.233	.134	.097	.077	.065	.056	.050	.045	.041	.037

TABLE 2, FOR TOP DISPLACEMENT UNDER SHEAR LOADING, FREE HEAD PILE

PAGE 4, POISSON'S RATIO = .40

LC/W \ L/LC =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
6	.655	.445	.355	.303	.267	.241	.222	.206	.193	.182
7	.618	.415	.329	.279	.246	.221	.203	.188	.176	.165
8	.587	.391	.308	.260	.228	.205	.187	.173	.162	.152
9	.559	.369	.290	.244	.213	.191	.174	.161	.150	.141
10	.536	.351	.274	.230	.201	.180	.163	.151	.140	.131
11	.515	.335	.260	.218	.190	.169	.154	.142	.132	.123
12	.496	.320	.248	.207	.180	.161	.146	.134	.124	.116
13	.479	.308	.238	.198	.172	.153	.138	.127	.118	.110
14	.463	.296	.228	.189	.164	.146	.132	.121	.112	.105
15	.449	.286	.219	.182	.157	.139	.126	.116	.107	.100
16	.436	.276	.211	.175	.151	.134	.121	.111	.102	.096
17	.424	.267	.204	.168	.145	.129	.116	.106	.098	.092
18	.412	.259	.197	.163	.140	.124	.112	.102	.094	.088
19	.402	.251	.191	.157	.135	.120	.108	.098	.091	.085
20	.392	.244	.185	.152	.131	.116	.104	.095	.088	.082
21	.383	.238	.180	.148	.127	.112	.101	.092	.085	.079
22	.375	.232	.175	.144	.123	.108	.097	.089	.082	.076
23	.367	.226	.170	.140	.119	.105	.094	.086	.079	.074
24	.359	.221	.166	.136	.116	.102	.092	.083	.077	.071
25	.352	.216	.162	.132	.113	.099	.089	.081	.075	.069
26	.345	.211	.158	.129	.110	.097	.087	.079	.073	.067
27	.338	.206	.154	.126	.107	.094	.084	.077	.071	.065
28	.332	.202	.151	.123	.105	.092	.082	.075	.069	.064
29	.326	.198	.148	.120	.102	.090	.080	.073	.067	.062
30	.320	.194	.145	.117	.100	.087	.078	.071	.065	.060
31	.315	.190	.142	.115	.098	.085	.076	.069	.064	.059
32	.310	.187	.139	.112	.095	.084	.075	.068	.062	.058
33	.305	.183	.136	.110	.093	.082	.073	.066	.061	.056
34	.300	.180	.133	.108	.092	.080	.071	.065	.059	.055
35	.295	.177	.131	.106	.090	.078	.070	.063	.058	.054
36	.291	.174	.129	.104	.088	.077	.068	.062	.057	.053
37	.287	.171	.126	.102	.086	.075	.067	.061	.056	.051
38	.283	.168	.124	.100	.085	.074	.066	.060	.054	.050
39	.279	.165	.122	.098	.083	.072	.065	.058	.053	.049
40	.275	.163	.120	.097	.082	.071	.063	.057	.052	.048
42	.267	.158	.116	.093	.079	.069	.061	.055	.050	.047
44	.261	.153	.113	.090	.076	.066	.059	.053	.049	.045
46	.254	.149	.109	.088	.074	.064	.057	.051	.047	.043
48	.248	.145	.106	.085	.071	.062	.055	.050	.045	.042
50	.242	.141	.103	.082	.069	.060	.053	.048	.044	.040
52	.237	.138	.100	.080	.067	.058	.052	.047	.043	.039
54	.232	.134	.098	.078	.065	.057	.050	.045	.041	.038
56	.227	.131	.095	.076	.064	.055	.049	.044	.040	.037
58	.222	.128	.093	.074	.062	.054	.047	.043	.039	.036
60	.218	.125	.091	.072	.060	.052	.046	.042	.038	.035

TABLE 2, FOR TOP DISPLACEMENT UNDER SHEAR LOADING, FREE HEAD PILE

PAGE 5, POISSON'S RATIO = .50

LC/W \ L/LC =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
6	.596	.405	.323	.276	.243	.220	.202	.187	.175	.165
7	.563	.378	.300	.254	.224	.202	.185	.171	.160	.150
8	.534	.356	.280	.237	.208	.187	.171	.158	.147	.138
9	.509	.336	.264	.222	.194	.174	.159	.147	.137	.128
10	.488	.320	.250	.209	.183	.164	.149	.137	.128	.120
11	.469	.305	.237	.198	.173	.154	.140	.129	.120	.112
12	.451	.292	.226	.189	.164	.146	.133	.122	.113	.106
13	.436	.280	.216	.180	.156	.139	.126	.116	.107	.100
14	.422	.270	.208	.172	.149	.133	.120	.110	.102	.095
15	.409	.260	.200	.165	.143	.127	.115	.105	.097	.091
16	.397	.251	.192	.159	.137	.122	.110	.101	.093	.087
17	.386	.243	.186	.153	.132	.117	.106	.097	.089	.083
18	.376	.236	.180	.148	.128	.113	.102	.093	.086	.080
19	.366	.229	.174	.143	.123	.109	.098	.090	.083	.077
20	.357	.223	.169	.139	.119	.105	.095	.086	.080	.074
21	.349	.217	.164	.135	.115	.102	.092	.084	.077	.072
22	.341	.211	.159	.131	.112	.099	.089	.081	.075	.069
23	.334	.206	.155	.127	.109	.096	.086	.078	.072	.067
24	.327	.201	.151	.124	.106	.093	.083	.076	.070	.065
25	.320	.196	.148	.120	.103	.090	.081	.074	.068	.063
26	.314	.192	.144	.117	.100	.088	.079	.072	.066	.061
27	.308	.188	.141	.115	.098	.086	.077	.070	.064	.060
28	.302	.184	.137	.112	.095	.084	.075	.068	.063	.058
29	.297	.180	.134	.109	.093	.082	.073	.066	.061	.056
30	.292	.177	.132	.107	.091	.080	.071	.065	.059	.055
31	.287	.173	.129	.105	.089	.078	.070	.063	.058	.054
32	.282	.170	.126	.102	.087	.076	.068	.062	.057	.052
33	.278	.167	.124	.100	.085	.074	.066	.060	.055	.051
34	.273	.164	.122	.098	.083	.073	.065	.059	.054	.050
35	.269	.161	.119	.096	.082	.071	.064	.058	.053	.049
36	.265	.158	.117	.095	.080	.070	.062	.056	.052	.048
37	.261	.156	.115	.093	.079	.069	.061	.055	.051	.047
38	.257	.153	.113	.091	.077	.067	.060	.054	.050	.046
39	.254	.151	.111	.089	.076	.066	.059	.053	.049	.045
40	.250	.148	.109	.088	.074	.065	.058	.052	.048	.044
42	.244	.144	.106	.085	.072	.062	.056	.050	.046	.042
44	.237	.140	.102	.082	.069	.060	.054	.048	.044	.041
46	.231	.136	.099	.080	.067	.058	.052	.047	.043	.039
48	.226	.132	.097	.077	.065	.057	.050	.045	.041	.038
50	.221	.129	.094	.075	.063	.055	.049	.044	.040	.037
52	.216	.125	.091	.073	.061	.053	.047	.042	.039	.036
54	.211	.122	.089	.071	.060	.052	.046	.041	.038	.035
56	.207	.120	.087	.069	.058	.050	.044	.040	.036	.034
58	.202	.117	.085	.067	.056	.049	.043	.039	.035	.033
60	.198	.114	.083	.066	.055	.048	.042	.038	.034	.032

TABLE 3, FOR TOP DISPLACEMENT (FIXED HEAD PILE)

PAGE 1, POISSON'S RATIO = .10

LC/W \ L/LC =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
6	.265	.180	.144	.122	.108	.119	.124	.125	.125	.118
7	.250	.168	.133	.113	.099	.109	.113	.115	.114	.107
8	.237	.158	.124	.105	.092	.101	.105	.106	.105	.099
9	.226	.149	.117	.099	.086	.094	.098	.098	.097	.091
10	.217	.142	.111	.093	.081	.088	.091	.092	.091	.085
11	.208	.135	.105	.088	.077	.083	.086	.086	.085	.080
12	.200	.130	.100	.084	.073	.079	.082	.082	.081	.076
13	.193	.124	.096	.080	.069	.075	.077	.078	.076	.072
14	.187	.120	.092	.077	.066	.072	.074	.074	.073	.068
15	.181	.115	.089	.073	.063	.068	.071	.071	.069	.065
16	.176	.112	.085	.071	.061	.066	.068	.068	.066	.062
17	.171	.108	.082	.068	.059	.063	.065	.065	.064	.059
18	.167	.105	.080	.066	.057	.061	.063	.062	.061	.057
19	.163	.102	.077	.064	.055	.059	.060	.060	.059	.055
20	.159	.099	.075	.062	.053	.057	.058	.058	.057	.053
21	.155	.096	.073	.060	.051	.055	.056	.056	.055	.051
22	.151	.094	.071	.058	.050	.053	.055	.054	.053	.049
23	.148	.091	.069	.056	.048	.052	.053	.052	.051	.048
24	.145	.089	.067	.055	.047	.050	.051	.051	.050	.046
25	.142	.087	.065	.053	.046	.049	.050	.049	.048	.045
26	.139	.085	.064	.052	.044	.047	.048	.048	.047	.044
27	.137	.083	.062	.051	.043	.046	.047	.047	.046	.042
28	.134	.082	.061	.050	.042	.045	.046	.046	.045	.041
29	.132	.080	.060	.049	.041	.044	.045	.044	.043	.040
30	.130	.078	.058	.047	.040	.043	.044	.043	.042	.039
31	.127	.077	.057	.046	.039	.042	.043	.042	.041	.038
32	.125	.075	.056	.045	.039	.041	.042	.041	.040	.037
33	.123	.074	.055	.045	.038	.040	.041	.040	.039	.036
34	.121	.073	.054	.044	.037	.039	.040	.039	.038	.036
35	.119	.071	.053	.043	.036	.038	.039	.039	.038	.035
36	.118	.070	.052	.042	.036	.038	.038	.038	.037	.034
37	.116	.069	.051	.041	.035	.037	.038	.037	.036	.033
38	.114	.068	.050	.040	.034	.036	.037	.036	.035	.033
39	.113	.067	.049	.040	.034	.036	.036	.036	.035	.032
40	.111	.066	.048	.039	.033	.035	.035	.035	.034	.031
42	.108	.064	.047	.038	.032	.034	.034	.034	.033	.030
44	.105	.062	.045	.037	.031	.033	.033	.032	.032	.029
46	.103	.060	.044	.035	.030	.031	.032	.031	.030	.028
48	.100	.059	.043	.034	.029	.030	.031	.030	.029	.027
50	.098	.057	.042	.033	.028	.030	.030	.029	.028	.026
52	.096	.056	.041	.032	.027	.029	.029	.028	.028	.025
54	.094	.054	.040	.032	.026	.028	.028	.028	.027	.025
56	.092	.053	.039	.031	.026	.027	.027	.027	.026	.024
58	.090	.052	.038	.030	.025	.026	.027	.026	.025	.023
60	.088	.051	.037	.029	.024	.026	.026	.025	.025	.023

TABLE 3, FOR TOP DISPLACEMENT (FIXED HEAD PILE)

PAGE 2, POISSON'S RATIO = .20

LC/W \ L/LC =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
6	.252	.172	.137	.117	.103	.113	.118	.120	.119	.112
7	.238	.160	.127	.108	.095	.104	.108	.109	.109	.102
8	.226	.151	.119	.100	.088	.096	.100	.101	.100	.094
9	.216	.142	.112	.094	.082	.090	.093	.094	.093	.087
10	.207	.135	.106	.089	.077	.084	.087	.088	.087	.081
11	.198	.129	.100	.084	.073	.079	.082	.082	.082	.076
12	.191	.124	.096	.080	.069	.075	.078	.078	.077	.072
13	.185	.119	.092	.076	.066	.072	.074	.074	.073	.068
14	.179	.114	.088	.073	.063	.068	.070	.070	.069	.065
15	.173	.110	.085	.070	.061	.065	.067	.067	.066	.062
16	.168	.106	.081	.067	.058	.063	.065	.064	.063	.059
17	.163	.103	.079	.065	.056	.060	.062	.062	.061	.057
18	.159	.100	.076	.063	.054	.058	.060	.059	.058	.054
19	.155	.097	.074	.061	.052	.056	.058	.057	.056	.052
20	.151	.094	.072	.059	.050	.054	.056	.055	.054	.050
21	.148	.092	.069	.057	.049	.052	.054	.053	.052	.049
22	.144	.089	.068	.055	.047	.051	.052	.052	.051	.047
23	.141	.087	.066	.054	.046	.049	.050	.050	.049	.046
24	.138	.085	.064	.052	.045	.048	.049	.049	.048	.044
25	.136	.083	.062	.051	.044	.047	.048	.047	.046	.043
26	.133	.081	.061	.050	.042	.045	.046	.046	.045	.042
27	.130	.080	.060	.049	.041	.044	.045	.045	.044	.040
28	.128	.078	.058	.047	.040	.043	.044	.043	.042	.039
29	.126	.076	.057	.046	.039	.042	.043	.042	.041	.038
30	.124	.075	.056	.045	.039	.041	.042	.041	.040	.037
31	.121	.073	.055	.044	.038	.040	.041	.040	.039	.036
32	.119	.072	.054	.043	.037	.039	.040	.039	.038	.036
33	.118	.071	.052	.042	.036	.038	.039	.039	.038	.035
34	.116	.069	.051	.042	.035	.037	.038	.038	.037	.034
35	.114	.068	.051	.041	.035	.037	.037	.037	.036	.033
36	.112	.067	.050	.040	.034	.036	.037	.036	.035	.032
37	.111	.066	.049	.039	.033	.035	.036	.035	.034	.032
38	.109	.065	.048	.039	.033	.035	.035	.035	.034	.031
39	.107	.064	.047	.038	.032	.034	.034	.034	.033	.031
40	.106	.063	.046	.037	.031	.033	.034	.033	.032	.030
42	.103	.061	.045	.036	.030	.032	.033	.032	.031	.029
44	.101	.059	.043	.035	.029	.031	.031	.031	.030	.028
46	.098	.058	.042	.034	.028	.030	.030	.030	.029	.027
48	.096	.056	.041	.033	.028	.029	.029	.029	.028	.026
50	.093	.055	.040	.032	.027	.028	.028	.028	.027	.025
52	.091	.053	.039	.031	.026	.027	.028	.027	.026	.024
54	.089	.052	.038	.030	.025	.027	.027	.026	.026	.023
56	.087	.051	.037	.029	.025	.026	.026	.026	.025	.023
58	.086	.049	.036	.029	.024	.025	.025	.025	.024	.022
60	.084	.048	.035	.028	.023	.024	.025	.024	.023	.022

TABLE 3, FOR TOP DISPLACEMENT (FIXED HEAD PILE)

PAGE 3, POISSON'S RATIO = .30

LC/W \ L/LC =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
6	.239	.162	.129	.110	.097	.107	.112	.113	.113	.106
7	.225	.151	.120	.102	.090	.098	.102	.103	.103	.097
8	.214	.142	.112	.095	.083	.091	.095	.095	.095	.089
9	.204	.135	.106	.089	.078	.085	.088	.089	.088	.082
10	.195	.128	.100	.084	.073	.080	.083	.083	.082	.077
11	.188	.122	.095	.079	.069	.075	.078	.078	.077	.072
12	.181	.117	.091	.076	.066	.071	.074	.074	.073	.068
13	.174	.112	.087	.072	.063	.068	.070	.070	.069	.065
14	.169	.108	.083	.069	.060	.065	.067	.067	.066	.061
15	.164	.104	.080	.066	.057	.062	.064	.064	.063	.058
16	.159	.101	.077	.064	.055	.059	.061	.061	.060	.056
17	.154	.097	.074	.061	.053	.057	.059	.058	.057	.054
18	.150	.094	.072	.059	.051	.055	.056	.056	.055	.051
19	.147	.092	.070	.057	.049	.053	.054	.054	.053	.049
20	.143	.089	.068	.056	.048	.051	.053	.052	.051	.048
21	.140	.087	.066	.054	.046	.050	.051	.050	.050	.046
22	.137	.085	.064	.052	.045	.048	.049	.049	.048	.045
23	.134	.082	.062	.051	.044	.047	.048	.047	.046	.043
24	.131	.080	.061	.049	.042	.045	.046	.046	.045	.042
25	.128	.079	.059	.048	.041	.044	.045	.045	.044	.041
26	.126	.077	.058	.047	.040	.043	.044	.043	.042	.039
27	.123	.075	.056	.046	.039	.042	.043	.042	.041	.038
28	.121	.074	.055	.045	.038	.041	.042	.041	.040	.037
29	.119	.072	.054	.044	.037	.040	.040	.040	.039	.036
30	.117	.071	.053	.043	.036	.039	.039	.039	.038	.035
31	.115	.069	.052	.042	.036	.038	.039	.038	.037	.034
32	.113	.068	.051	.041	.035	.037	.038	.037	.036	.034
33	.111	.067	.050	.040	.034	.036	.037	.036	.036	.033
34	.109	.066	.049	.039	.033	.035	.036	.036	.035	.032
35	.108	.064	.048	.039	.033	.035	.035	.035	.034	.031
36	.106	.063	.047	.038	.032	.034	.035	.034	.033	.031
37	.105	.062	.046	.037	.031	.033	.034	.033	.033	.030
38	.103	.061	.045	.036	.031	.033	.033	.033	.032	.029
39	.102	.060	.044	.036	.030	.032	.033	.032	.031	.029
40	.100	.059	.044	.035	.030	.031	.032	.031	.031	.028
42	.098	.058	.042	.034	.029	.030	.031	.030	.029	.027
44	.095	.056	.041	.033	.028	.029	.030	.029	.028	.026
46	.093	.054	.040	.032	.027	.028	.029	.028	.027	.025
48	.090	.053	.039	.031	.026	.027	.028	.027	.027	.024
50	.088	.052	.038	.030	.025	.027	.027	.026	.026	.024
52	.086	.050	.037	.029	.025	.026	.026	.026	.025	.023
54	.085	.049	.036	.028	.024	.025	.025	.025	.024	.022
56	.083	.048	.035	.028	.023	.024	.025	.024	.023	.022
58	.081	.047	.034	.027	.023	.024	.024	.023	.023	.021
60	.079	.046	.033	.026	.022	.023	.023	.023	.022	.020

TABLE 3, FOR TOP DISPLACEMENT (FIXED HEAD PILE)

PAGE 4, POISSON'S RATIO = .40

LC/W \ L/LC =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
6	.223	.151	.121	.103	.091	.100	.104	.106	.105	.099
7	.210	.141	.112	.095	.084	.091	.095	.096	.096	.090
8	.199	.133	.105	.088	.078	.085	.088	.089	.088	.083
9	.190	.126	.099	.083	.073	.079	.082	.083	.082	.077
10	.182	.119	.093	.078	.068	.074	.077	.077	.076	.072
11	.175	.114	.089	.074	.065	.070	.072	.073	.072	.067
12	.169	.109	.084	.070	.061	.066	.069	.069	.068	.064
13	.163	.105	.081	.067	.058	.063	.065	.065	.064	.060
14	.157	.101	.077	.064	.056	.060	.062	.062	.061	.057
15	.153	.097	.075	.062	.053	.058	.059	.059	.058	.055
16	.148	.094	.072	.059	.051	.055	.057	.057	.056	.052
17	.144	.091	.069	.057	.049	.053	.055	.054	.054	.050
18	.140	.088	.067	.055	.048	.051	.053	.052	.051	.048
19	.137	.086	.065	.053	.046	.049	.051	.050	.050	.046
20	.133	.083	.063	.052	.044	.048	.049	.049	.048	.044
21	.130	.081	.061	.050	.043	.046	.047	.047	.046	.043
22	.127	.079	.060	.049	.042	.045	.046	.046	.045	.042
23	.125	.077	.058	.047	.041	.043	.044	.044	.043	.040
24	.122	.075	.056	.046	.039	.042	.043	.043	.042	.039
25	.120	.073	.055	.045	.038	.041	.042	.042	.041	.038
26	.117	.072	.054	.044	.037	.040	.041	.040	.040	.037
27	.115	.070	.053	.043	.036	.039	.040	.039	.038	.036
28	.113	.069	.051	.042	.036	.038	.039	.038	.037	.035
29	.111	.067	.050	.041	.035	.037	.038	.037	.036	.034
30	.109	.066	.049	.040	.034	.036	.037	.036	.036	.033
31	.107	.065	.048	.039	.033	.035	.036	.036	.035	.032
32	.105	.063	.047	.038	.032	.035	.035	.035	.034	.031
33	.104	.062	.046	.037	.032	.034	.034	.034	.033	.031
34	.102	.061	.045	.037	.031	.033	.034	.033	.032	.030
35	.100	.060	.045	.036	.030	.032	.033	.032	.032	.029
36	.099	.059	.044	.035	.030	.032	.032	.032	.031	.029
37	.097	.058	.043	.035	.029	.031	.032	.031	.030	.028
38	.096	.057	.042	.034	.029	.030	.031	.031	.030	.027
39	.095	.056	.041	.033	.028	.030	.030	.030	.029	.027
40	.093	.055	.041	.033	.028	.029	.030	.029	.029	.026
42	.091	.054	.039	.032	.027	.028	.029	.028	.027	.025
44	.089	.052	.038	.031	.026	.027	.028	.027	.027	.024
46	.086	.051	.037	.030	.025	.026	.027	.026	.026	.024
48	.084	.049	.036	.029	.024	.026	.026	.025	.025	.023
50	.082	.048	.035	.028	.024	.025	.025	.025	.024	.022
52	.081	.047	.034	.027	.023	.024	.024	.024	.023	.021
54	.079	.046	.033	.027	.022	.023	.024	.023	.022	.021
56	.077	.045	.032	.026	.022	.023	.023	.023	.022	.020
58	.076	.044	.032	.025	.021	.022	.022	.022	.021	.020
60	.074	.043	.031	.025	.021	.022	.022	.021	.021	.019

TABLE 3, FOR TOP DISPLACEMENT (FIXED HEAD PILE)

PAGE 5, POISSON'S RATIO = .50

LC/W \ L/LC =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
6	.203	.138	.110	.094	.083	.091	.095	.096	.096	.090
7	.191	.129	.102	.086	.076	.083	.087	.088	.087	.082
8	.182	.121	.095	.081	.071	.077	.080	.081	.080	.075
9	.173	.114	.090	.076	.066	.072	.075	.075	.075	.070
10	.166	.109	.085	.071	.062	.068	.070	.070	.070	.065
11	.159	.104	.081	.067	.059	.064	.066	.066	.065	.061
12	.153	.099	.077	.064	.056	.060	.062	.063	.062	.058
13	.148	.095	.074	.061	.053	.057	.059	.059	.059	.055
14	.143	.092	.071	.059	.051	.055	.057	.057	.056	.052
15	.139	.088	.068	.056	.049	.052	.054	.054	.053	.050
16	.135	.085	.065	.054	.047	.050	.052	.052	.051	.047
17	.131	.083	.063	.052	.045	.048	.050	.050	.049	.045
18	.128	.080	.061	.050	.043	.047	.048	.048	.047	.044
19	.124	.078	.059	.049	.042	.045	.046	.046	.045	.042
20	.121	.076	.057	.047	.041	.043	.045	.044	.044	.041
21	.119	.074	.056	.046	.039	.042	.043	.043	.042	.039
22	.116	.072	.054	.044	.038	.041	.042	.041	.041	.038
23	.113	.070	.053	.043	.037	.040	.040	.040	.039	.037
24	.111	.068	.051	.042	.036	.038	.039	.039	.038	.035
25	.109	.067	.050	.041	.035	.037	.038	.038	.037	.034
26	.107	.065	.049	.040	.034	.036	.037	.037	.036	.033
27	.105	.064	.048	.039	.033	.035	.036	.036	.035	.032
28	.103	.063	.047	.038	.032	.035	.035	.035	.034	.032
29	.101	.061	.046	.037	.032	.034	.034	.034	.033	.031
30	.099	.060	.045	.036	.031	.033	.034	.033	.032	.030
31	.098	.059	.044	.036	.030	.032	.033	.032	.032	.029
32	.096	.058	.043	.035	.030	.031	.032	.032	.031	.029
33	.094	.057	.042	.034	.029	.031	.031	.031	.030	.028
34	.093	.056	.041	.033	.028	.030	.031	.030	.029	.027
35	.091	.055	.041	.033	.028	.029	.030	.030	.029	.027
36	.090	.054	.040	.032	.027	.029	.029	.029	.028	.026
37	.089	.053	.039	.032	.027	.028	.029	.028	.028	.026
38	.087	.052	.038	.031	.026	.028	.028	.028	.027	.025
39	.086	.051	.038	.030	.026	.027	.028	.027	.027	.025
40	.085	.050	.037	.030	.025	.027	.027	.027	.026	.024
42	.083	.049	.036	.029	.024	.026	.026	.026	.025	.023
44	.081	.048	.035	.028	.024	.025	.025	.025	.024	.022
46	.079	.046	.034	.027	.023	.024	.024	.024	.023	.021
48	.077	.045	.033	.026	.022	.023	.024	.023	.023	.021
50	.075	.044	.032	.026	.021	.023	.023	.022	.022	.020
52	.073	.043	.031	.025	.021	.022	.022	.022	.021	.019
54	.072	.042	.030	.024	.020	.021	.022	.021	.020	.019
56	.070	.041	.030	.024	.020	.021	.021	.021	.020	.018
58	.069	.040	.029	.023	.019	.020	.020	.020	.019	.018
60	.067	.039	.028	.022	.019	.020	.020	.019	.019	.017

TABLE 4, PERCENTAGE CORRECTIONS FOR TABLES 2 AND 3

LC/W	ABSOLUTE VALUE OF (L/LC-1)						
	.0	.05	.10	.15	.20	.25	.30
2	3	2	1	0	0	0	0
4	4	2	1	1	0	0	0
6	4	3	2	1	0	0	0
8	4	3	2	1	0	0	0
10	5	3	2	1	0	0	0
12	5	4	2	1	0	0	0
14	6	4	2	1	0	0	0
16	6	4	2	1	0	0	0
18	7	4	3	1	0	0	0
20	7	5	3	1	0	0	0
22	7	5	3	1	0	0	0
24	8	5	3	2	0	0	0
26	8	6	3	2	0	0	0
28	9	6	4	2	1	0	0
30	9	6	4	2	1	0	0
32	10	7	4	2	1	0	0
34	10	7	4	2	1	0	0
36	11	7	4	2	1	0	0
38	11	7	5	2	1	0	0
40	11	8	5	2	1	0	0
42	12	8	5	3	1	0	0
44	12	8	5	3	1	0	0
46	13	9	5	3	1	0	0
48	13	9	6	3	1	0	0
50	14	9	6	3	1	0	0
52	14	10	6	3	1	0	0
54	14	10	6	3	1	0	0
56	15	10	6	3	1	0	0
58	15	10	7	3	1	0	0
60	16	11	7	4	1	0	0
62	16	11	7	4	1	0	0
64	17	11	7	4	1	0	0
66	17	12	7	4	1	0	0
68	17	12	7	4	1	0	0
70	18	12	8	4	2	0	0

TABLE 5, FOR OBTAINING TOP DISPLACEMENT (MOMENT LOADING) AND TOP SLOPES, FREE HEAD PILE

LC/W \ L/LC =	.10	.20	.30	.40	.50	.60	.70	.80	.90	.51
6	3.338	1.594	1.084	0.846	0.710	0.622	0.561	0.516	0.481	0.699
7	2.908	1.389	0.944	0.737	0.618	0.542	0.488	0.449	0.419	0.609
8	2.575	1.230	0.836	0.652	0.547	0.480	0.433	0.398	0.371	0.539
9	2.311	1.104	0.750	0.586	0.491	0.430	0.388	0.357	0.333	0.484
10	2.096	1.001	0.680	0.531	0.446	0.390	0.352	0.324	0.302	0.439
11	1.918	0.916	0.623	0.486	0.408	0.357	0.322	0.296	0.277	0.402
12	1.767	0.844	0.574	0.448	0.376	0.329	0.297	0.273	0.255	0.370
13	1.639	0.783	0.532	0.415	0.348	0.305	0.275	0.253	0.236	0.343
14	1.528	0.730	0.496	0.387	0.325	0.285	0.257	0.236	0.220	0.320
15	1.431	0.683	0.464	0.362	0.304	0.266	0.240	0.221	0.206	0.300
16	1.345	0.643	0.437	0.341	0.286	0.251	0.226	0.208	0.194	0.282
17	1.270	0.606	0.412	0.322	0.270	0.236	0.213	0.196	0.183	0.266
18	1.202	0.574	0.390	0.304	0.255	0.224	0.202	0.186	0.173	0.252
19	1.141	0.545	0.370	0.289	0.243	0.213	0.192	0.176	0.165	0.239
20	1.086	0.519	0.353	0.275	0.231	0.202	0.182	0.168	0.157	0.227
21	1.036	0.495	0.336	0.262	0.220	0.193	0.174	0.160	0.149	0.217
22	0.991	0.473	0.322	0.251	0.211	0.184	0.166	0.153	0.143	0.207
23	0.949	0.453	0.308	0.240	0.202	0.177	0.159	0.147	0.137	0.199
24	0.910	0.435	0.296	0.231	0.194	0.170	0.153	0.141	0.131	0.191
25	0.875	0.418	0.284	0.222	0.186	0.163	0.147	0.135	0.126	0.183
26	0.842	0.402	0.273	0.213	0.179	0.157	0.141	0.130	0.121	0.176
27	0.812	0.388	0.264	0.206	0.173	0.151	0.136	0.125	0.117	0.170
28	0.784	0.374	0.254	0.199	0.167	0.146	0.132	0.121	0.113	0.164
29	0.757	0.362	0.246	0.192	0.161	0.141	0.127	0.117	0.109	0.159
30	0.733	0.350	0.238	0.186	0.156	0.136	0.123	0.113	0.106	0.153
31	0.710	0.339	0.230	0.180	0.151	0.132	0.119	0.110	0.102	0.149
32	0.688	0.329	0.223	0.174	0.146	0.128	0.116	0.106	0.099	0.144
33	0.668	0.319	0.217	0.169	0.142	0.124	0.112	0.103	0.096	0.140
34	0.648	0.310	0.211	0.164	0.138	0.121	0.109	0.100	0.094	0.136
35	0.630	0.301	0.205	0.160	0.134	0.117	0.106	0.097	0.091	0.132
36	0.613	0.293	0.199	0.155	0.130	0.114	0.103	0.095	0.088	0.128
37	0.597	0.285	0.194	0.151	0.127	0.111	0.100	0.092	0.086	0.125
38	0.582	0.278	0.189	0.147	0.124	0.108	0.098	0.090	0.084	0.122
39	0.567	0.271	0.184	0.144	0.120	0.106	0.095	0.088	0.082	0.119
40	0.553	0.264	0.180	0.140	0.118	0.103	0.093	0.085	0.080	0.116
41	0.540	0.258	0.175	0.137	0.115	0.101	0.091	0.083	0.078	0.113
42	0.527	0.252	0.171	0.134	0.112	0.098	0.089	0.081	0.076	0.110
43	0.515	0.246	0.167	0.130	0.109	0.096	0.087	0.080	0.074	0.108
44	0.504	0.241	0.163	0.128	0.107	0.094	0.085	0.078	0.073	0.105
45	0.493	0.235	0.160	0.125	0.105	0.092	0.083	0.076	0.071	0.103
46	0.482	0.230	0.156	0.122	0.102	0.090	0.081	0.074	0.070	0.101
47	0.472	0.225	0.153	0.120	0.100	0.088	0.079	0.073	0.068	0.099
48	0.462	0.221	0.150	0.117	0.098	0.086	0.078	0.071	0.067	0.097
49	0.453	0.216	0.147	0.115	0.096	0.084	0.076	0.070	0.065	0.095
50	0.444	0.212	0.144	0.112	0.094	0.083	0.075	0.069	0.064	0.093
51	0.435	0.208	0.141	0.110	0.093	0.081	0.073	0.067	0.063	0.091
52	0.427	0.204	0.139	0.108	0.091	0.080	0.072	0.066	0.062	0.089
53	0.419	0.200	0.136	0.106	0.089	0.078	0.070	0.065	0.060	0.088
54	0.412	0.197	0.134	0.104	0.087	0.077	0.069	0.064	0.059	0.086
55	0.404	0.193	0.131	0.102	0.086	0.075	0.068	0.062	0.058	0.085
56	0.397	0.190	0.129	0.101	0.084	0.074	0.067	0.061	0.057	0.083
57	0.390	0.186	0.127	0.099	0.083	0.073	0.066	0.060	0.056	0.082
58	0.384	0.183	0.125	0.097	0.082	0.071	0.064	0.059	0.055	0.080
59	0.377	0.180	0.122	0.096	0.080	0.070	0.063	0.058	0.054	0.079
60	0.371	0.177	0.120	0.094	0.079	0.069	0.062	0.057	0.053	0.078

TABLE 6, FOR FIXING MOMENTS

LC/W \ L/LC =	.10	.20	.30	.40	.50	.51
6	0.34	0.71	1.05	1.34	1.60	1.63
7	0.39	0.82	1.20	1.54	1.84	1.87
8	0.44	0.92	1.36	1.74	2.08	2.11
9	0.49	1.03	1.51	1.94	2.31	2.35
10	0.54	1.14	1.67	2.14	2.55	2.59
11	0.59	1.24	1.83	2.34	2.79	2.83
12	0.64	1.35	1.98	2.54	3.03	3.07
13	0.69	1.45	2.14	2.74	3.26	3.31
14	0.74	1.56	2.29	2.94	3.50	3.55
15	0.79	1.66	2.45	3.13	3.74	3.79
16	0.84	1.77	2.60	3.33	3.97	4.03
17	0.90	1.87	2.76	3.53	4.21	4.27
18	0.95	1.98	2.91	3.73	4.45	4.52
19	1.00	2.09	3.07	3.93	4.69	4.76
20	1.05	2.19	3.22	4.13	4.92	5.00
21	1.10	2.30	3.38	4.33	5.16	5.24
22	1.15	2.40	3.53	4.53	5.40	5.48
23	1.20	2.51	3.69	4.73	5.63	5.72
24	1.25	2.61	3.84	4.93	5.87	5.96
25	1.30	2.72	4.00	5.13	6.11	6.20
26	1.35	2.82	4.16	5.32	6.35	6.44
27	1.40	2.93	4.31	5.52	6.58	6.68
28	1.45	3.04	4.47	5.72	6.82	6.92
29	1.50	3.14	4.62	5.92	7.06	7.16
30	1.55	3.25	4.78	6.12	7.30	7.40
31	1.60	3.35	4.93	6.32	7.53	7.65
32	1.65	3.46	5.09	6.52	7.77	7.89
33	1.70	3.56	5.24	6.72	8.01	8.13
34	1.75	3.67	5.40	6.92	8.24	8.37
35	1.80	3.77	5.55	7.12	8.48	8.61
36	1.85	3.88	5.71	7.31	8.72	8.85
37	1.90	3.99	5.86	7.51	8.96	9.09
38	1.95	4.09	6.02	7.71	9.19	9.33
39	2.00	4.20	6.18	7.91	9.43	9.57
40	2.05	4.30	6.33	8.11	9.67	9.81
42	2.16	4.51	6.64	8.51	10.14	10.29
44	2.26	4.72	6.95	8.91	10.62	10.78
46	2.36	4.94	7.26	9.31	11.09	11.26
48	2.46	5.15	7.57	9.70	11.57	11.74
50	2.56	5.36	7.88	10.10	12.04	12.22
52	2.66	5.57	8.19	10.50	12.52	12.70
54	2.76	5.78	8.51	10.90	12.99	13.18
56	2.86	5.99	8.82	11.30	13.46	13.67
58	2.96	6.20	9.13	11.69	13.94	14.15
60	3.06	6.41	9.44	12.09	14.41	14.63

CHAPTER FIVE

POINT FORCE IN A NON-UNIFORM ELASTIC HALF-SPACE

5.1 INTRODUCTION

In Chapter 3 an analysis was presented which considered a laterally loaded pile to be situated in a half space with constant elastic properties. It is desirable to improve on this half-space as a soil model and one improvement which may be made is to allow the elastic properties to vary within the space. Chapter 3 was based on Mindlin's solutions for a point force in a constant elastic half-space: this chapter presents a solution for the point force in an elastic half-space the shear modulus of which varies with depth. A perturbation expansion is used and Mindlin's solutions are utilized as Green's functions which allow the solution to be expressed in integral form. The integration is carried out numerically.

5.2 THEORY

The equations of equilibrium at a point in space may be written as

$$\sigma_{ij,j} + f_i = 0 \ . \tag{5.1}$$

If the position of the point is given in rectangular Cartesian Coordinates by $\underline{x} = (x_1, x_2, x_3)$ and the only force applied is a point force at $\underline{y} = (y_1, y_2, y_3)$ then the body force term in equation (5.1) can be written as

$$f_i = P_i \delta(\underline{x} - \underline{y}) \quad , \tag{5.2}$$

where $\delta(\underline{x} - \underline{y})$ is a Dirac delta-function. The stress-displacement relations at a point in a linearly elastic medium with shear modulus G and Poisson's ratio μ are:

$$\sigma_{ij} = G\,(u_{i,j} + u_{j,i} + \frac{2\mu}{1-2\mu} u_{k,k}\,\delta_{ij}), \tag{5.3}$$

where δ_{ij} is the Kronecker delta. This may be written, defining function F, as $\sigma_{ij} = G\, F_{ij}\,(\underline{u})$. (5.4)

If the half-space is defined as that region where $x_3 \geqslant 0$, then the boundary conditions will be

$$\sigma_{i3} = 0 \text{ on } x_3 = 0 \quad .$$

That is,

$$F_{i3}\,(\underline{u}) = 0 \text{ on } x_3 = 0 \tag{5.5}$$

No restrictions have yet been placed on G or μ : equations (5.1) and (5.3) are expressions at a point. However, we shall now consider the case in which μ is constant throughout the half-space and G varies in the x_3 direction in the manner

$$G = G\,(x_3) = G_o\,(1 + \epsilon . h\,(x_3)). \tag{5.6}$$

Substitution from (5.3) into (5.1) gives

$$G_o\,(1 + \epsilon . h\,(x_3))\,(u_{i,jj} + \frac{1}{1-2\mu} u_{j,ji})$$

$$+ G_o\,\epsilon\, h_{,3}\,(x_3)\, F_{i3}\,(\underline{u}) + f_i = 0. \tag{5.7}$$

If ϵ is small, a perturbation solution can be developed by writing the displacement $\underline{u}$ in the form

$$u_i = u_i^{(o)} + \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \ldots \tag{5.8}$$

(5.8) may now be substituted into (5.7); but as equations (5.7) are true for any value of ϵ, the coefficients of powers of ϵ must each be zero. The zero-order equations are

$$G_o \left(u^{(o)}_{i,jj} + \frac{1}{1-2\mu} u^{(o)}_{j,ji}\right) + f_i = 0 \quad , \tag{5.9}$$

with boundary conditions, from (5.5), of

$$F_{i3} (\underline{u}^{(o)}) = 0 \text{ on } x_3 = 0.$$

These are the equations solved by Mindlin for a uniform elastic half-space, and the displacements $u_i^{(o)}$ are his solutions.

The first-order equations are

$$G_o \left(u^{(1)}_{i,jj} + \frac{1}{1-2\mu} u^{(1)}_{j,ji}\right) + G_o\, h(x_3) \left(u^{(o)}_{i,jj} + \frac{1}{1-2\mu} u^{(o)}_{j,ji}\right)$$

$$+ G_o\, h_{,3} (x_3)\, F_{i3} (\underline{u}^{(o)}) = 0 \quad . \tag{5.10}$$

which may be rewritten, using (5.9), as

$$G_o \left(u^{(1)}_{i,jj} + \frac{1}{1-2\mu} u^{(1)}_{j,ji}\right) - h (x_3)\, f_i$$

$$+ G_o\, h_{,3} (x_3)\, F_{i3} (\underline{u}^{(o)}) = 0 \quad . \tag{5.11}$$

If ϵ is small enough that

$$u_i \approx u_i^{(o)} + \epsilon u_i^{(1)} \quad , \tag{5.12}$$

then the problem becomes essentially that of solving equations (5.11).

The notation for displacements is now changed by writing

$$\underline{u} = \underline{u}^{(o)}$$

and $$\underline{v} = \underline{u}^{(1)} .$$

Equations (5.9) thus become

$$G_o \left(u_{i,jj} + \frac{1}{1-2\mu} u_{j,ji}\right) + f_i = 0 .$$

Mindlin has solved these equations by writing, in effect,

$$u_i = \sum u_{ik} , \tag{5.13}$$

and solving

$$G_o \left(u_{ik,jj} + \frac{1}{1-2\mu} u_{jk,ji}\right) + \delta_{ik} \; f_i = 0, \tag{5.14}$$

where the summation convention does not apply on the last term and where u_{ik} may be interpreted as the displacement at $\underline{x}$ in direction i due to a point force at $\underline{y}$ in direction k. (5.14) may be written more simply, using (5.2) with $P_i = 1$, as

$$L_{ij} \; u_{jk} \; (\underline{x},\underline{y}) + \delta_{ik} \; \delta \; (\underline{x}-\underline{y}) = 0 , \tag{5.15}$$

which defines a matrix of differential operators L_{ij}. Equations (5.11), which are to be solved, may likewise be written

$$L_{ij} \; v_{jk} \; (\underline{x},\underline{y}) - h \; (x_3) \; \delta_{ik} \; \delta \; (\underline{x} - \underline{y})$$

$$+ G_o \; h_{,3} \; (x_3) \; F_{i3} \; (u_k \; (\underline{x},\underline{y}) \;) = 0 . \tag{5.16}$$

Now any equation of the form

$$L_{ij} \; v_j \; (\underline{x},\underline{y}) + \rho_i \; (\underline{x},\underline{y}) = 0 \tag{5.17}$$

may be written

$$v_j \; (\underline{x},\underline{y}) = - \int_{\underline{z}} \rho_i \; (\underline{z},\underline{y}) \; g_{ji} \; (\underline{x},\underline{z}) \; d\underline{z} , \tag{5.18}$$

where g_{ji} are Green's functions.

Thus (5.15) may be written

$$u_{jk}\ (\underline{x},\underline{y}) = -\int_{\underline{z}} \delta_{ik}\,\delta\,(\underline{z} - \underline{y})\ g_{ji}\ (\underline{x},\underline{z})\ d\underline{z}$$

$$= -\ g_{jk}\ (\underline{x},\underline{y}), \qquad (5.19)$$

and the required Green's functions are

$$g_{ji}\ (\underline{x},\underline{z}) = -\ u_{ji}\ (\underline{x},\underline{z})\ . \qquad (5.20)$$

The solutions to (5.16) may now be written

$$v_{jk}\ (\underline{x},\underline{y}) = \int_{\underline{z}} \Big[G_o h_{,3}\ (z_3)\ F_{i3}\ (u_k\ (\underline{z},\underline{y})\ )$$

$$-\ h\ (z_3)\ \delta_{ik}\ \delta\,(\underline{z}-\underline{y}) \Big]\ u_{ji}\ (\underline{x},\underline{z})\ d\underline{z}\ . \qquad (5.21)$$

Or, writing

$$\phi_{ik}\ (\underline{x},\underline{y}) = G_o h_{,3}\ (x_3)\ F_{i3}\ (u_k\ (\underline{x},\underline{y})\ ), \qquad (5.22)$$

$$v_{jk}\ (\underline{x},\underline{y}) = \int_{\underline{z}} \phi_{ik}\ (\underline{z},\underline{y})\ u_{ji}\ (\underline{x},\underline{z})\ d\underline{z}$$

$$-\ h(y_3)\ u_{jk}\ (\underline{x},\underline{y}). \qquad (5.23)$$

The problem now becomes one of evaluating the integral in this equation, and this is discussed in the next section.

The mathematical assumptions made may now be examined. They are

1. $u_i = u_i^{(o)} + \epsilon u_i^{(1)}$ is a sufficiently good approximation to u_i ((5.12)).
2. G varies according to $G = G_o\ (1 + \epsilon\,.h(x_3)\ )$.
3. μ is constant.

Assumption 1

u_i was in fact written (equation (5.8)) as

$$u_i = u_i^{(o)} + \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \ldots \epsilon^r u_i^{(r)} + \ldots$$

If this is substituted in (5.7), and coefficients of ϵ^r equated to zero, the equations obtained are

$$G_o \left(u_{i,jj}^{(r)} + \frac{1}{1-2\mu} u_{j,ji}^{(r)}\right) + G_o h(x_3) \left(u_{i,jj}^{(r-1)} + \frac{1}{1-2\mu} u_{j,ji}^{(r-1)}\right)$$
$$+ G_o h_{,3}(x_3) F_{i3}(\underline{u}^{(r-1)}) = 0 .$$

Thus terms up to any r can be obtained as long as the preceding term is known as a function of $\underline{x}$ and $\underline{y}$.

In fact the assumption should be valid if ϵ is small, there being no reason to suspect $u_i^{(2)}$ to be substantially greater than $u_i^{(1)}$. It should be noted here that small ϵ does not imply an insignificant variation of G: consider the case $h(x_3) = x_3$ and $\epsilon = .01$, that is, $G(x_3) = G_o (1 + .01x_3)$. It will be seen that $G(100) = 2G_o$ which is a far from insignificant variation if effects are to be considered at depths of this order.

Assumption 2

G has been assumed to vary with x_3 only. This is not necessary, as allowing G to vary with x_i will not affect the form of the analysis. A completely general form of variation could be considered.

Assumption 3

It is also not necessary that μ be constant. If equations (5.3) are written

$$\sigma_{ij} = G\,(u_{i,j} + u_{j,i} + \nu .\, u_{k,k}\,\delta_{ij}) \quad ,$$

where $\nu = \frac{2}{1-2\mu}$, and ν is allowed to vary in a manner similar to G in (5.6) then equations similar to (5.7) can be obtained. The zero-order equations are once again those solved by Mindlin, and the analysis may be carried out in a manner similar to that presented. It will be noticed that ν becomes infinite when $\mu = 0.5$. This infinite term does of course occur also in equation (5.3), but the term cancels and the Mindlin equations do not contain an infinite term when $\mu = 0.5$. However, in the case of μ varying, there may be a problem if $\mu_o = 0.5$ and ν_o becomes infinite.

5.3 EVALUATION OF THE INTEGRAL

5.3.1 Introduction

It is necessary for the solution of (5.23) to evaluate the integral

$$I\,(\underline{x},\underline{y}) = \int_{\underline{z}} \phi_{ik}\,(\underline{z},\underline{y})\; u_{ji}\,(\underline{x},\underline{z})\; d\underline{z} \quad .$$

Mindlin has expressed his solution $u(\underline{x},\underline{y})$ in a form such that the origin of coordinates lies in the free surface and such that $y_1 = y_2 = 0$, and it will be convenient for $V\,(\underline{x},\underline{y})$ to be expressed in the same form. $\phi\,(\underline{z},\underline{y})$ is a function of $u(\underline{z},\underline{y})$ and Mindlin's solutions may be used directly with $\underline{z}$ substituted for $\underline{x}$. However, for $u(\underline{x},\underline{z})$ direct substitution into the Mindlin solutions will relate $\underline{x}$ and $\underline{z}$ to a wrong origin, and so a coordinate shift is necessary. $u(\underline{x},\underline{z})$ may be obtained by writing:

$$x_1 - z_1 \text{ for } x_1$$

$$x_2 - z_2 \text{ for } x_2$$

$$x_3 - z_3 \text{ for } x_3$$

and $$z_3 \text{ for } y_3$$

in the Mindlin expressions for $u(\underline{x},\underline{y})$.

In the evaluation of the integral some singularities will occur, for the integrand will be infinite when $\underline{z} = \underline{y}$ in ϕ and when $\underline{z} = \underline{x}$ in u. These singularities occur because there is a finite force acting on an infinitely small area. In fact the force will act on a finite area and so it will be physically meaningful to carry out the integration by excluding small regions surrounding $\underline{z} = \underline{x}$ and $\underline{z} = \underline{y}$: an analytic solution for the integral can be easily obtained using this technique if the indefinite integral is known.

Such a solution has not been attempted because of the magnitude of effort involved: the integrand, when expanded, contains over 200 terms, and as it is a volume integral, over 600 terms must be integrated. The first of these terms (omitting constants) is

$$\frac{z_3 - y_3}{R_1 \, S_1^{\,3}}$$

where

$$R_1 = \left((x_1 - z_1)^2 + (x_2 - z_2)^2 + (x_3 - z_3)^2 \right)^{\frac{1}{2}}$$

and $$S_1 = (z_1^2 + z_2^2 + (z_3 - y_3)^2)^{\frac{1}{2}} ,$$

so that the individual terms are themselves not easily integrated. Instead, a numerical integration, using the digital computer, has been carried out.

This has the disadvantage that the integral cannot now be expressed as a function of $\underline{x}$ and $\underline{y}$, meaning that a second-order perturbation term cannot be obtained, and also that an explicit expression for the stress state at any point cannot be given: stresses now have to be obtained from a difference form of equations (5.3).

5.3.2 Methods of Numerical Integration

Two methods of numerical integration were used: the trapezoidal rule and the Monte Carlo method.

The trapezoidal rule method evaluates the integrand on a regularly spaced grid of points (three-dimensional in this case), averages the values, and multiplies the average by the volume of integration. Implicit in the method is the assumption that the integrand varies linearly between grid points. Obviously, if the volume of integration is large and the grid spacing is not too great, a large number of points will need to be considered. Singular points can be treated either by ensuring that the points are not selected, or by giving to the integrand a value equal to its value nearby. To reduce the chance of a systematic error, two grids were taken and the results averaged: one grid has points which lie on the coordinate axes and the other has points which straddle the axes symmetrically.

The Monte Carlo method uses a random number process to evaluate the integral. Consider a function of x, $0 < f(x) < 1$ in $0 < x < 1$. $\int_0^1 f(x)\,dx$ is equal to the probability that a randomly selected point in the area defined by the above limits lies beneath the curve. This probability may be obtained by randomly selecting x, evaluating f(x), and averaging the values obtained. It would seem that in a case where a large number of points need to be considered this method may provide a quicker solution than the trapezoidal rule. Also, the possibility of a systematic error developing is eliminated.

Random numbers were selected using the subprogram RANDU supplied by IBM for the 360/44 in their Scientific Subroutines Package, the distribution of random numbers obtained being checked and found to be very satisfactory. The subroutine requires one initial random number and it generates a "string" of following ones from this. Three different random numbers were generated, one for each coordinate: this was considered preferable to using three numbers from the same string, as the latter process would exclude the legitimate possibility of two or three coordinates being equal. The value of the integral is of course dependent on the values of the initial random numbers, but it can be expected that after a large number of points have been considered the integral values will be essentially equal.

The expanded form of the integrand and program listings for the evaluation of the integral by both methods are contained in Appendix IV. The rate of computation was about 5000 points/minute.

5.3.3 Results of numerical integration

Figs 5.1 - 5.3 show some results of the numerical integration. All results are for the horizontal displacement at (0,0,20) due to a horizontal point force of magnitude 100,000 at (0,0,10) in the half-space with G_0 = 1000 and μ = .45. The three figures are intended to provide (1) a comparison between the trapezoidal and Monte Carlo methods, (2) a consideration of the effect of the range of integration (where to say that the range is R means that the integration is performed over a cube of side R), and (3) a useable value of the integral.

Fig. 5.1 gives trapezoidal rule solutions for ranges of 100 and 10,000 and some Monte Carlo solutions for ranges of 100, 10,000, and 1,000,000. Two facts emerge: the trapezoidal rule appears superior for the two lower ranges; and the effect of the range of integration is significant. The Monte Carlo method is still

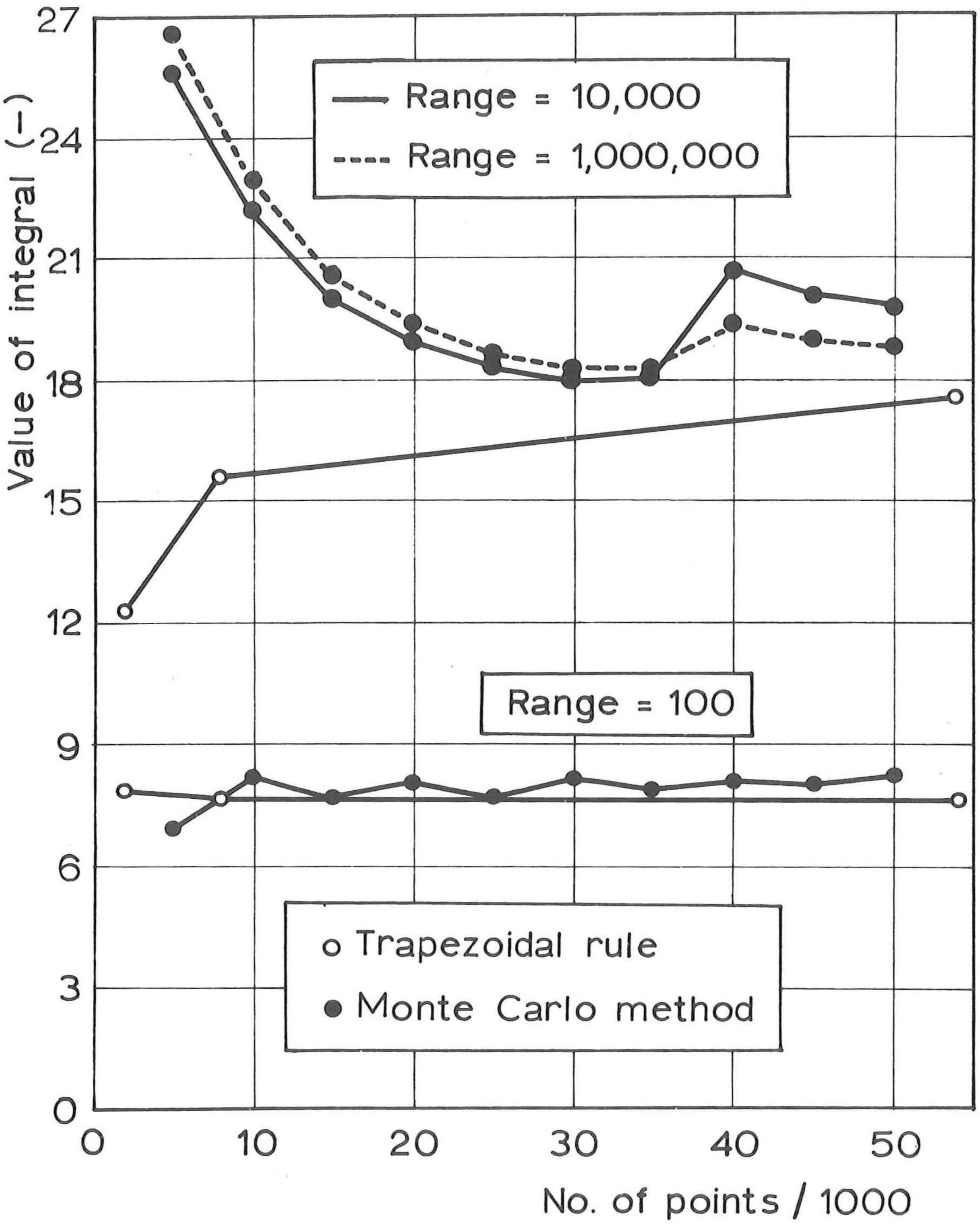


FIG. 5.1 - INTEGRATION BY NUMERICAL METHODS.

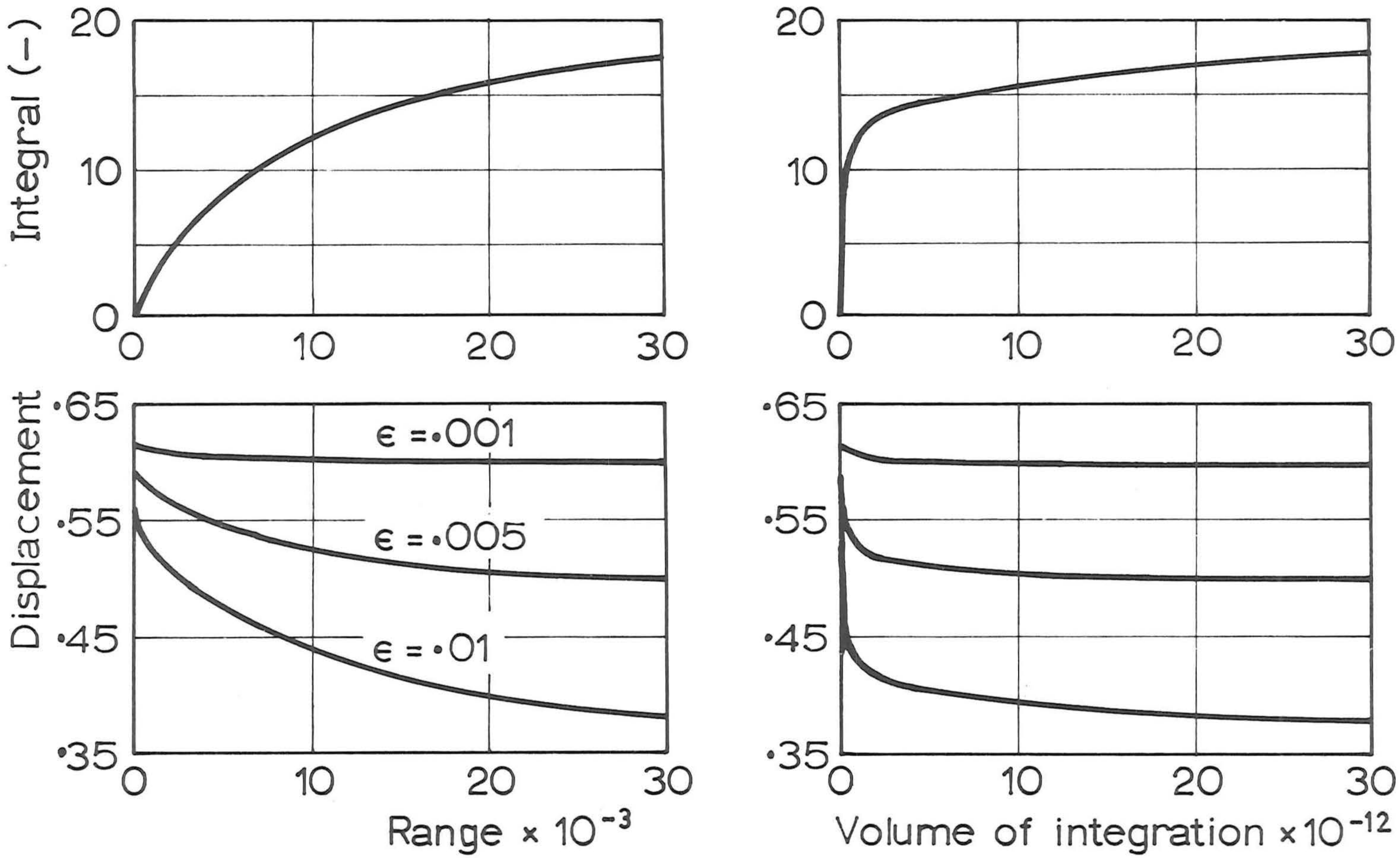


FIG. 5.2 - EFFECT OF RANGE OF INTEGRATION.

at the stage of being dependant on the initial random numbers, and 50,000 points is evidently too small a number to give reliable results, although these particular selections of initial numbers provide a reasonable correlation between the two methods. Extension of these graphs gave the following results for a range of 10,000: the trapezoidal rule gave, using 128,000 points, a value for the integral of -18.9 while the Monte Carlo method, using 150,000 points, gave -18.3.

The effect of the range of integration is further studied in fig. 5.2, which gives values obtained from the trapezoidal rule, the point spacing being kept constant as the range varies. The top curves show how the value of the integral varies with the range and also with the volume of integration, which is a direct measure of the work involved in carrying out the integration. The bottom curves show how the range affects the displacement. In this case $h(z_3) = z_3$, so that G varies linearly with depth. The given values of ϵ appear small but it has already been pointed out that such values are significant. It appears from fig. 5.2 that a range of 30,000 is not high enough and it has already been shown that for a range of 10,000 the value of the integral from the trapezoidal rule was still increasing after taking 128,000 points; thus it seems that to take a large enough range but to evaluate the integral without excessive work will require the use of the Monte Carlo method.

Fig. 5.3 shows an evaluation of the integral over range 1,000,000 using the Monte Carlo method. The particular selection of initial random numbers used has pointed out clearly a danger of the method. Between the 80,000th and the 100,000th points selected, one point has given a very high value for the integrand. The point was isolated, and although its selection was quite legitimate, such a selection should occur only once in 8,000,000 times. This point will distort the true value of the integral after only 700,000 points have been considered, and so a plot is also given which neglects it.

A summary of the results obtained for the integration is set out in table 5.1. Values are given with the high point considered and not considered.

Method	Range	No. of points taken	Value
Trapezoidal	100	54,000	-7.8
Trapezoidal	10,000	128,000	-18.9
Monte Carlo	10,000	150,000	-18.3
Monte Carlo	1,000,000	700,000	-21.0, -27.3

Table 5.1 Value of integral

The three items which the figures were intended to

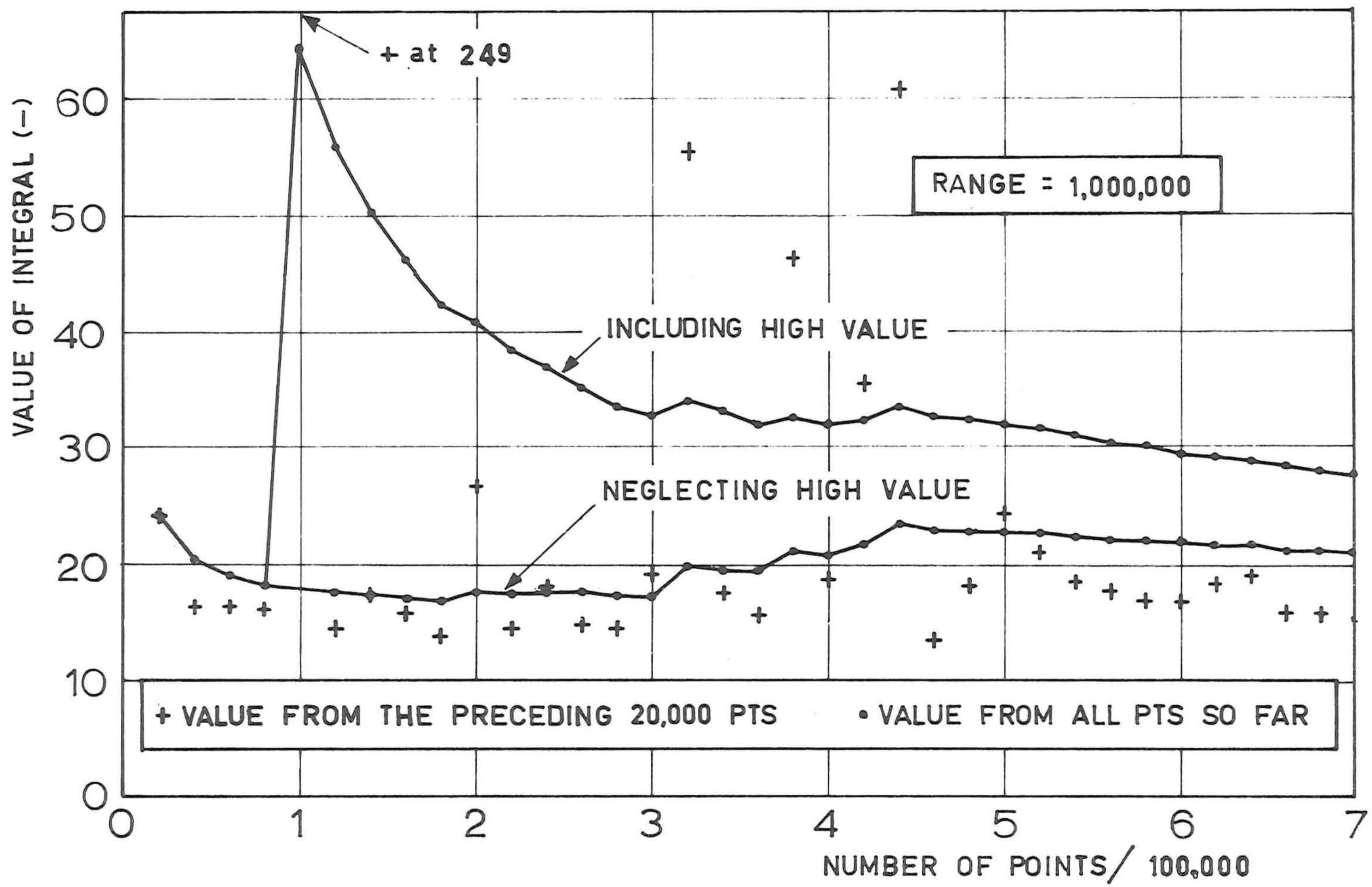


FIG. 5.3 - EVALUATION OF INTEGRAL BY MONTE CARLO METHOD.

examine may now be discussed.

(1) A comparison between the two methods shows that for small ranges of integration the trapezoidal rule is superior, but in this case the range must be large and the Monte Carlo method is probably preferable. Perhaps a better method would to be use the trapezoidal rule with a closer spacing of points near the centre of the range. The Monte Carlo method appears to be useful for volume integrals, but in cases where the integrand peaks within the range of integration , care must be used in the application of the method.

(2) The effect of changing the range of integration has been shown to be significant even when the range is large. The integration should be performed throughout the half-space but it might be expected that effects at some distance from the point force would have little influence on the results. This has not proved to be the case and the question arises as to what is the physical significance of the integration range. The case considered, which uses a range of 1,000,000, implies that the effects of a point force at depth 20 feet depend on conditions 100 miles distant.

(3) On consideration of the table of results it seems that a value for the integral of -23.0 may be a reasonable estimate. This value may be used to compute the displacement:

assuming $h(z_3) = z_3$

$$v = I - h(y_3) . u$$
$$= -23.0 - 6.2$$
$$= -29.2$$

so, letting $\epsilon = .01$, displacement $= u + \epsilon v$

$$= .621 - .01 \times 29.2$$
$$= .329$$

It is interesting to note that the variation of G considered has produced a reduction in displacement of almost 50%, showing that the very high values of G below the point have had considerable effect.

5.4 Discussion and Conclusions

An analysis has been presented which provides a solution for a point force acting in a half-space with elastic properties which may vary with position in a quite general manner. The solution has necessitated the evaluation of a volume integral which contains a large number of quite complex terms, and the integration has been performed numerically.

The numerical integration provides a number of disadvantages which have been mentioned, the chief of which is that only two terms in equation (5.8) can be considered. This means that the variation in properties must be both slow (though it may be far from insignificant) and continuous. A real soil may have sudden variations in properties because of stratification, and the analysis given should allow step functions to be used. However, it will then become complex and high order terms in equations (5.8) will surely need to be considered. Further disadvantages arising from the numerical integration include the large amount of computation time required (many point forces would need to be considered for a laterally loaded pile solution), and the rather unsatisfactory nature of the obtained solution (its approximate nature, and the uncertainty about the significance of the range of integration).

Because of these reasons, a pile solution has not been obtained from the general theory contained in this chapter. Instead, another type of solution has been developed which is approximate but which allows, in practice, a more general variation of soil properties and which is less time consuming. It is presented in the next chapter.

CHAPTER SIX

LATERALLY LOADED PILE IN A LAYERED ELASTIC HALF-SPACE

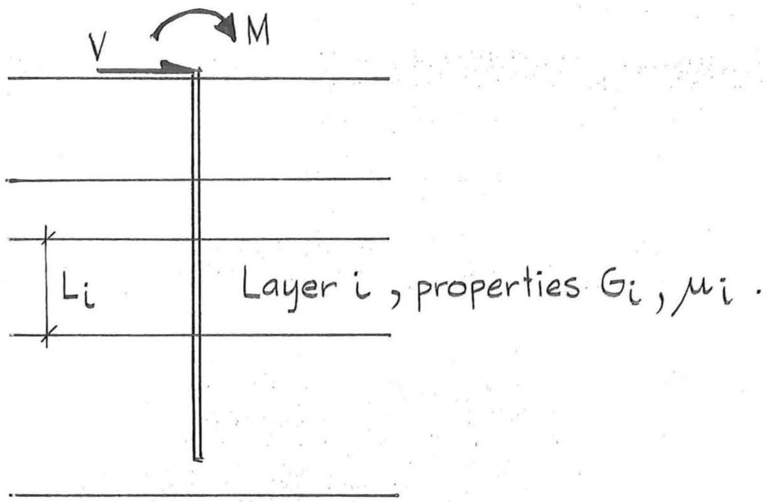
6.1 INTRODUCTION

The intention in this chapter is to modify the soil model used by allowing the elastic half-space to have properties which vary with depth in a stepped manner. The idea of a layered soil is useful both because real soil is often layered and because any form of continuous variation may be approximately represented if a number of thin layers is used.

The approach adopted was to make a fundamental assumption which allowed an approximate solution to be developed and then to refine the model so that the solution became more accurate. The basis of the solution is the analysis presented in Chapter 3 for a uniform elastic half-space (this analysis now becoming the special case of a soil with one layer) so this chapter may be considered as an extension of Chapter 3.

6.2 THEORY

It is required to solve the system represented by



6.2.1 Basic Assumption

6.2.1.1 Initial Basic Assumption

An initial attempt at solution made the following assumption.

Consider the pile to be separated into portions, the length of each being the thickness of the layer in which it is positioned. The loading on and displaced shape of each portion will be such as to preserve internal equilibrium and compatibility of the complete pile, but will be found assuming each portion to be situated in an elastic half-space with the properties of the appropriate layer. That is, a pile element of length L_i is assumed to be situated at the position of layer i in a half-space with properties G_i and μ_i. This means that there will be some lack of compatibility at the boundaries between layers: the displacements of layer i at its boundaries will not match those of neighbouring layers. The model can be expected to be reasonably accurate if layers are not too thin and changes in properties are not too large.

This essentially finite element approach can be very simply generalised so that the pile also has varying properties: each pile element is bounded by changes in either pile or soil properties.

A computer program was developed for a pile in a layered soil using the basic assumption outlined above and the detailed theory behind the program is included in Appendix V.

6.2.1.2 Final Basic Assumption

Having produced a solution using the initial basic assumption it was decided to make an improvement on the assumption. It is obvious that reaction forces outside layer i will affect the displacements within this layer. The initial assumption has produced a banded soil flexibility matrix, and an improvement would be to provide non-zero terms off the band by accounting in some way for the effects of forces in other layers. Thus it was required to approximately compute the displacement at a point in layer i due to a force in layer j, and it was decided that the best way to do this would be to assume soil properties of G_i and μ_i.

But it now became obvious that to make the improvement was to make the different basic assumption that the displacement at a point depends on the elastic properties at only that point. In this light it was decided to abandon the finite element representation of the pile and to use the new fundamental assumption by making a simple extension to the theory of Chapter 3. Thus the basic analysis of a laterally loaded pile in a layered elastic half-space is as for a constant elastic half-space but where Mindlin's expressions for displacement use as elastic properties those values which exist at the point of displacement.

It should be noted that the adoption of this basic assumption does not imply the use of a Winkler-type model: the continuity of the foundation is preserved and the effective foundation modulus k_h will still depend on pile as well as soil properties. That is, to say the displacement is dependent on soil properties at the point of displacement is not to say it is directly proportional to them.

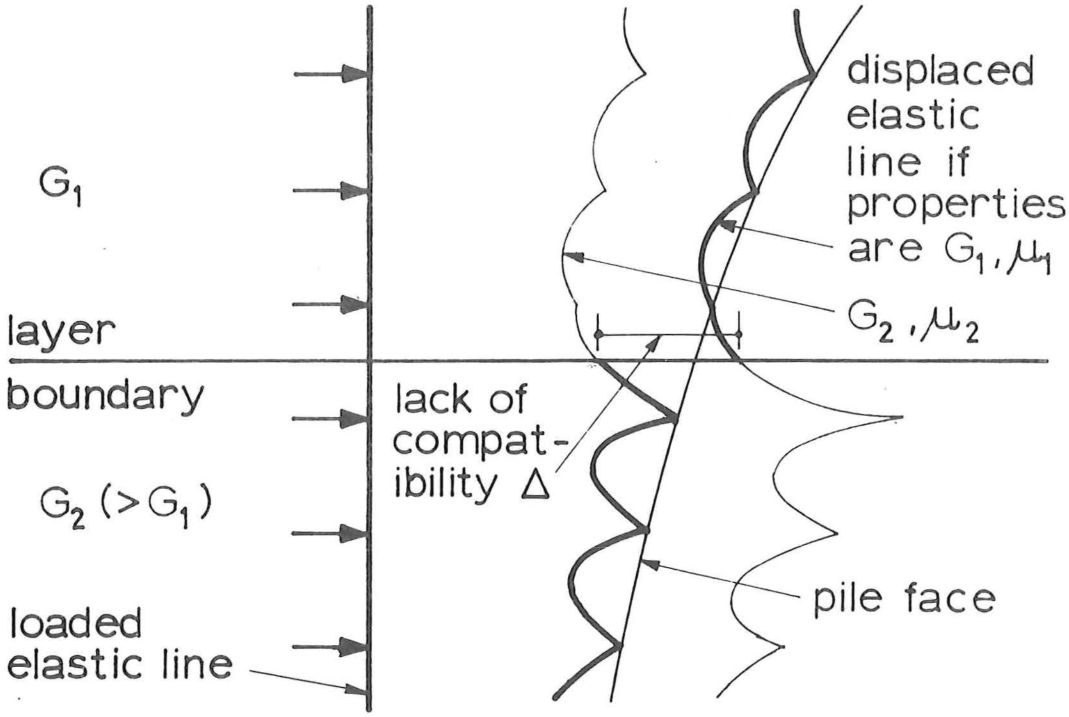
6.2.2 Improvement on the Basic Assumption

Although the basic assumption does not now require a layered half-space but may consider directly any variation in properties, the idea of soil layers is retained in order than an improvement may be made on this basis.

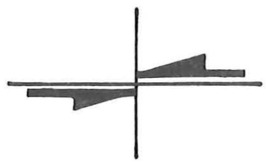
Thus there are a number of soil layers, each acting as if surrounding layers had identical properties: soil layer i acts as if the whole half-space had properties G_i and μ_i, but layer (i + 1) acts as if the medium had properties G_{i+1} and μ_{i+1}. Incompatibilities will occur at the boundaries between layers, and the model can be improved by applying at the boundaries restraints which tend to restore compatibility. An exact solution could be achieved by applying some set of equal and opposite forces at each boundary so that compatibility is completely preserved, but this would be a difficult process in practice so it was decided to choose only the most likely maximum incompatibility and correct this. Whether further corrections were necessary could be later decided.

Accordingly, it was decided to correct the lack of displacement compatibility occurring between layers in a horizontal direction at the pile face.

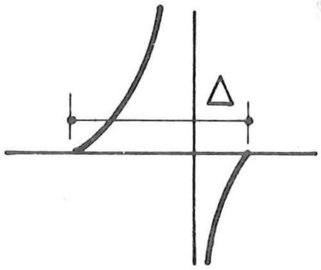
Fig. 6.1 will help to clarify how this incompatibility occurs. The elastic line referred to in the figure is a soil elastic line as distinct from the pile elastic line which is also shown. The force system acting on the soil would produce different displacements in the half-



Situation after initial solution using basic assumption.



correcting forces



shape of elastic line produced by correcting forces.

FIG. 6.1 - INCOMPATIBILITY OF DISPLACEMENTS AT LAYER BOUNDARY

space with properties G_1 and μ_1 from those in the half-space with G_2 and μ_2. Compatibility of the elastic line is restored by applying sets of equal and opposite forces to each side of the boundary.

It is obvious that the changed horizontal soil displacements which occur because of the correcting forces will not fit the pile equations. Thus the system has to be solved again with the pile-soil compatibility equations changed from

$$u_{pile} = u_{soil}$$

to

$$u_{pile} = u_{soil} + u_c,$$

where u_c are the displacements due to the correcting forces.

This second solution will give different values for the soil reaction forces and so provide different layer boundary incompatibilities, different correcting forces, and different values for u_c. The final solution is thus arrived at iteratively, with the iteration being stopped when the differences between successive solutions are small.

The steps involved in arriving at a final solution are represented in block diagram form in fig. 6.2.

6.2.3 Effect of Compatibility Correction

It is now necessary to decide whether further improvements should be made to the model. Specifically, it has to be decided whether compatibility at layer boundaries should be restored at other positions and in

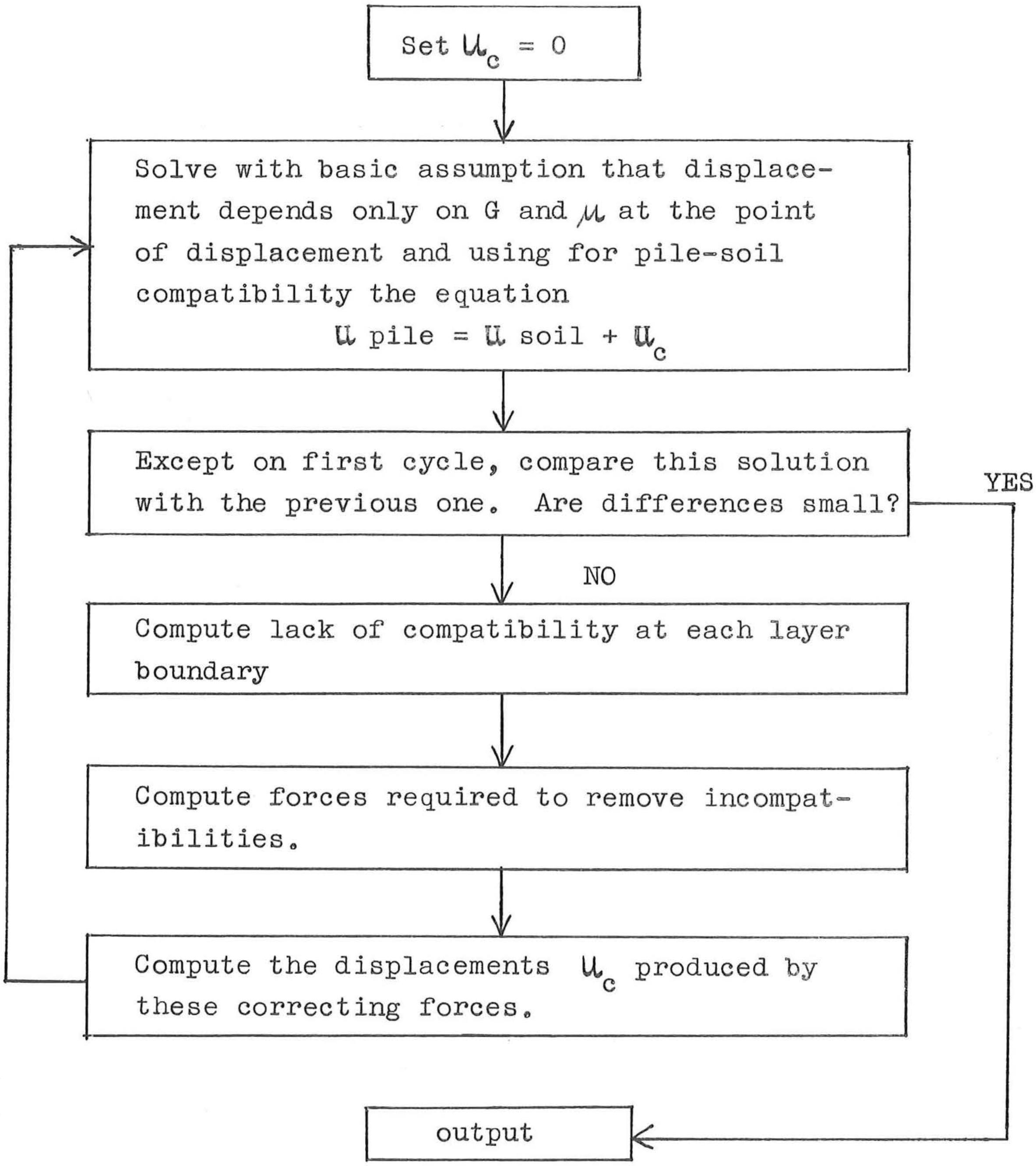


Fig. 6.2 - Solution of a pile in a layered soil

other directions.

Typical effects on the ground level displacement and slope for some two-layer cases are given in Table 6.1. The pile used is a medium length one ($L = L_c$ if the soil has uniform properties G_1 and μ_1), with shear and moment loading, and the ratios given represent (value after correction / value before correction). It should perhaps be emphasised that Table 6.1 does not compare a single layer case with a two layer one: it compares the initial solution for the two layer case with the final one.

Case	G_2/G_1	Boundary depth	Slope ratio	Displacement ratio
1	10	.25L	.92	.90
2	10	.75L	1.00	.99
3	.1	.25L	1.09	1.04
4	.1	.75L	.98	.97

Table 6.1 - Effects of Compatibility Correction

It will be seen that the correction effect can be significant: it will be especially so if the change in properties is large or if the layer boundary is close to the soil surface. However, the effect is not especially large and the computation time which would be necessary to further improve the soil model is probably not justified by the improvement it would provide. Such further improvements may in any case be only illusory, for it is not entirely clear what would happen away from the pile face at a layer boundary in a real soil.

Accordingly, no further compatibility corrections are made.

6.3 THE COMPUTER PROGRAM

The computer was programmed according to the basic block diagram contained in the last section.

Convergence appears to occur always, and is quite rapid, depending on the number of layers and on the sizes of variations in G and μ. The criteria for ending cycling are based on the pile top and bottom displacements, comparing values with those of the previous cycle. If top displacements differ by 1% or less, and bottom displacements by 5% or less, then cycling stops. As convergence is rapid, given values for these quantities will in fact be much closer to the limiting values than 1% and 5% respectively.

Different criteria for top and bottom displacements are used for two reasons: values at the pile top are of more interest, and they are generally of greater magnitude, so that absolute differences are more comparable. Two layer cases normally require two correcting cycles (sometimes only one), and three layer cases usually require three. Rapidity of convergence has been improved by automatically adjusting layer boundary positions so that they lie midway between force points, this shift producing little effect on the final result if a reasonably large number of force points is taken.

Input requirements are the same as those described in Chapter 3, except of course that G and μ must be read for each layer, and the boundary positions (which may be below the pile tip) must be specified. Computer storage capacity has limited the allowed number of layers to ten. Output is similar to that described in Chapter 3.

A complete phase overlay structure (explained in Appendix III) has been used to minimise storage requirements, the compatibility correction process forming one complete phase which may be called from, and which returns control to, either phase FREE or phase FIXED. These are the phases mentioned in Chapter 3, which are now extended to include the fundamental assumption of this chapter.

A program listing is included in Appendix III.

6.4 SOME SOLUTIONS

6.4.1 Some typical two-layer systems.

Fig. 6.3 shows the same pile as was used to obtain table 6.1, with both moment and shear loading at ground surface level. It gives the displaced shape of the pile if the half-space has properties G_1 and μ_1 and

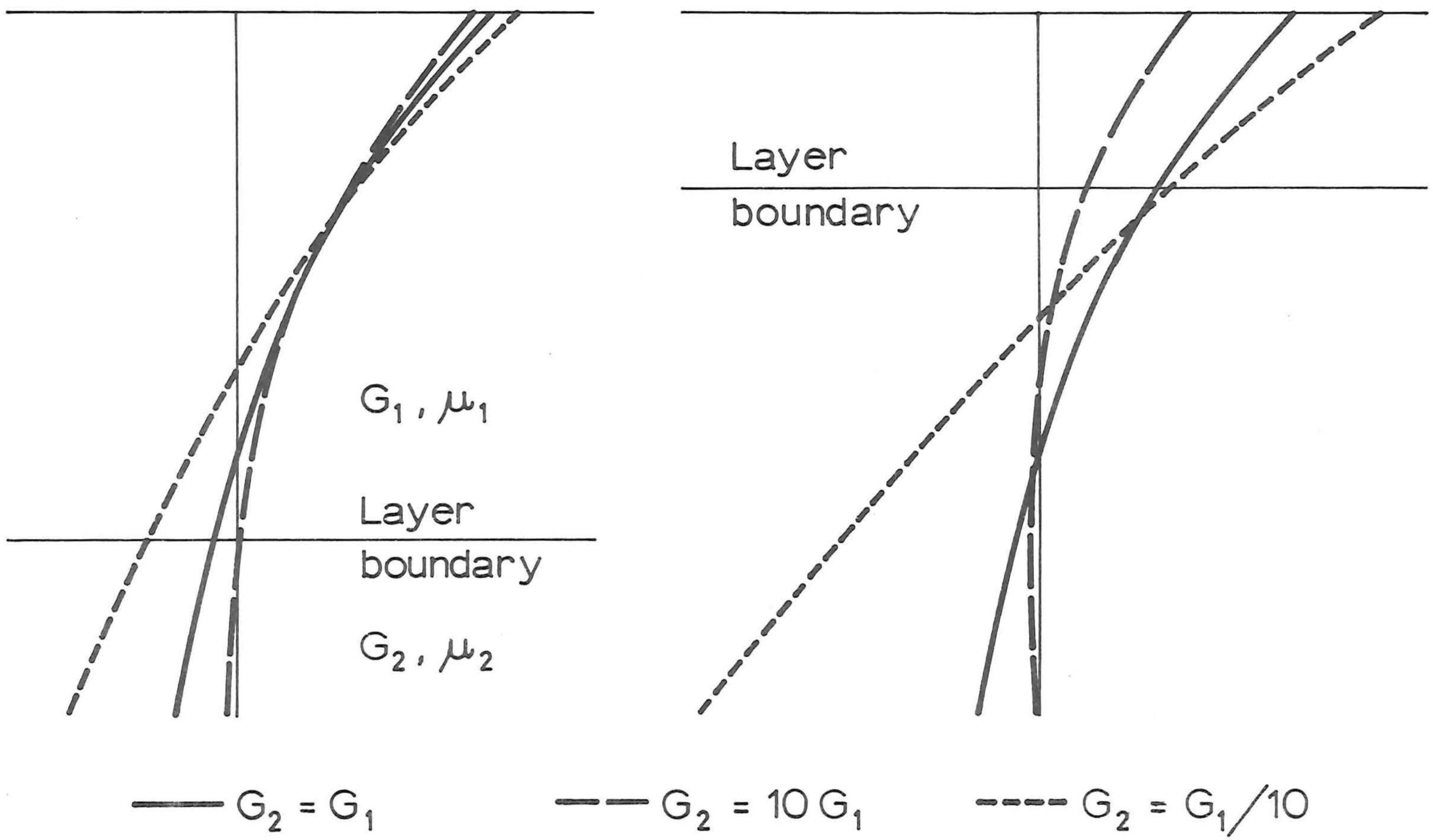


FIG. 6.3 - PILE DISPLACED SHAPES FOR SOME 2-LAYER SOILS.

introduces a second soil layer which has a shear modulus G_2 with a value of firstly $10G_1$ and secondly $G_1/10$.

The effects of the second soil layer are much as expected, with the stiff layer tending to fix the pile near the bottom and the flexible layer allowing a relatively large displacement at the pile tip.

A noteworthy fact is that the relatively large variations in G produce only a limited effect on the top displacement and slope. Further examples tried by the writer lend weight to the statement that these quantities depend largely on the soil properties to a depth of about $\frac{1}{2}$ L_c.

Another interesting observation is that the stiff bottom layer causes higher pile curvatures: maximum bending moments are compared in table 6.2, in which all moment values are expressed as a proportion of that obtained for a pile in a soil of constant properties.

Case	G_2/G_1	Boundary Depth	Maximum B.M.
1	1	-	1.00
2	10	.25L	1.16
3	10	.75L	1.02
4	.1	.25L	.94
5	.1	.75L	.99

Table 6.2 - Maximum bending moments for some 2-layer Systems.

6.4.2 Fitting a Test Result

It is worthwhile to show that a solution can in fact be obtained which fits some published test data. There has sometimes been a tendency to justify the use of the Winkler soil model because variations of soil modulus can be found which allow such a fit.

The test used for this purpose is that carried out by Kerisel and Adam[43] which was mentioned in Chapter 3. Fig. 3.10 shows the result of choosing a soil of constant properties (G = .41 t/sq.cm, $\mu = 0.5$) which gave the same top displacement. The theoretical deflected shape was somewhat different from that reported. By using trial and error to find the number of layers, the boundary positions and the properties, it was found that to use a three layer soil with boundaries at 45 and 95 cm and values for G of .25, .75 and .14 t/sq.cm respectively (μ being 0.5 in all cases) gave a reasonably good fit down the whole length of the pile. A deflected shape, if drawn, would fit closely that given in fig. 3.10.

This rather unusual distribution suggests consolidation of soil near the surface but subsequent loosening or perhaps inelastic behaviour of the top 17-18in.

The interesting fact to emerge is that, contrary to Poulos' suggestion ([7] p.20), the negative displacements recorded appear to have been due to a lower value of G at depth.

6.5 CONCLUSION

A theory has been developed from which a computer program was written to give an approximate solution for a laterally loaded pile in an elastic isotropic half space with properties which vary with depth, in layers.

The fundamental assumption is made that the displacement at a point in the space is dependent on the soil properties at only that point, and the solution obtained using this assumption is improved upon by partially correcting the lack of soil compatibility which occurs at the boundaries between layers. That no further compatibility correction is necessary is deduced from the results obtained, which show the basic assumption to provide a reasonable first approximation to the solution.

Some typical results have illustrated the fact that variations in soil properties at depths below about $\frac{1}{2}L_c$ do not very greatly affect the pile top displacement or slope.

CHAPTER SEVEN

STRESSES IN A LAYERED ELASTIC HALF-SPACE

7.1 INTRODUCTION

Studies of the behaviour of a laterally loaded pile have not so far required a knowledge of the stresses within the soil. Such knowledge, however, would probably be of use if extensions to the theory were to be made. As has already been noted, the ability to determine stresses is a significant advantage of treating the soil as an elastic continuum: it is the first soil model to recognise the simple fact that an understanding of soil behaviour is essential if a laterally loaded pile-soil system is to be rationally analysed. Following are some of the probable uses to which a knowledge of the stresses might be put.

An elastic stress distribution would appear to be the best starting point for a study of soil behaviour which is not linearly elastic.

A study of laterally loaded pile groups would almost certainly require elastic stress distributions because of the property of superposition: stress systems for single piles under lateral and axial loads may be super-imposed to give a total stress system, and yield criteria or other modifications may then be applied to the total system.

Consideration of dynamic effects may require a knowledge of the soil stress state.

Because of these reasons, the stresses due to a laterally loaded pile in a layered elastic half-space have been found.

An additional reason may also be given. Chapter 8 describes a photoelastic investigation intended to check the stresses, and by implication the deflections, in a layered system. Thus a knowledge of the stresses has an immediate use in allowing a check of the whole analysis, and in particular the assumptions of Chapter 6.

7.2 THEORY

7.2.1 General

Mindlin has given expressions for the stresses due to a point force in a constant elastic half-space. These expressions, together with the deflection analysis of Chapter 6, form the basis for the solution of the stress state at a point.

The soil model of the previous chapter is also used here, so that the stresses at a point are calculated using the elastic properties at the point, and the force system includes the correcting forces at the layer boundaries.

There is some difficulty associated with describing meaningfully a three dimensional stress state, especially graphically. The difficulty lies not only with the means of presenting data, but also with the choosing of the quantities to be represented. The quantity to be represented will depend on the use to which the stress distribution is to be put. For example, an examination of yielding would require a description consistent with the yield criterion used. A two dimensional stress distribution could be described by stress trajectories or perhaps contours of maximum principal or shear stress, but a three dimensional representation of this type would be most difficult to present.

Three methods of describing the stress state at a point are used here, and others may be derived from them. Mindlin's expressions give the direct and shear stresses on the planes perpendicular to the three axes which describe the half-space. The first method of description simply gives these six values at a point. Another method of description, which uses stress differences, is described in the next chapter. It is especially suitable for comparisons with photoelastic work. The third gives the three principal stresses and the three direction cosines associated with each of them.

7.2.2 The Finding of Principal Stresses

Numerous authors (for example, Southwell[44]) have shown that if p is a principal stress with direction cosines l, m, and n, then

$$\left.\begin{aligned} l\,(\sigma_{11} - p) + m\,\sigma_{12} + n.\,\sigma_{13} &= 0 \\ l.\,\sigma_{21} + m\,(\sigma_{22} - p) + n.\sigma_{23} &= 0 \\ l.\,\sigma_{31} + m\,\sigma_{32} + n\,(\sigma_{33} - p) &= 0 \end{aligned}\right\} \qquad (7.1)$$

Consider this to be a set of homogeneous simultaneous equations in l, m and n. Then as these quantities cannot vanish simultaneously,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} \sigma_{11} - p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - p \end{vmatrix} = 0 \qquad (7.2)$$

which is a cubic equation in p with three real roots. It may be written

$$p^3 - I_1 p^2 - I_2\, p - I_3 = 0 \qquad (7.3)$$

where I_1, I_2, I_3 are the three stress invariants:

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \sigma_{23}^{\,2} + \sigma_{31}^{\,2} + \sigma_{12}^{\,2} - \sigma_{22}.\,\sigma_{33} - \sigma_{33}.\,\sigma_{11} - \sigma_{11}.\,\sigma_{22}$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

Equation (7.3) may be solved numerically using

Newton's law which states that if x is an approximation to a root of the polynomial f(p) then a better approximation is given by the term

$$-\frac{f(p)}{f'(p)}$$

. Once the root p_1 has been found in this way the other two may be obtained using

$$p_2, p_3 = \tfrac{1}{2}\left(-a \pm \sqrt{a^2-4b}\right)$$

where $a = I_1 + p_1$

and $b = I_2 + ap_1$

The direction cosines of any principal stress may now be found from equation (7.1). If the cofactor of a_1 (equation (7.2)) is written A_1, then in general, solutions to (7.1) are given by

$$l = kA_1 \ , \ m = kB_1 \ , \ n = kC_1 \ .$$

The further relationship

$$l^2 + m^2 + n^2 = 1$$

enables k to be found.

If $p = \sigma_{22}$ or σ_{33}

then $A_1 = B_1 = C_1 = 0$

but in these cases the solutions are

$$l = 0 \ , \ m = 1, \ n = 0$$

and

$$l = 0 \ , \ m = 0, \ n = 1$$

7.3 THE COMPUTER PROGRAM

The separate program phase STRESS, which is listed in Appendix III, may be called to carry out the calculations indicated in this chapter.

Input obtained directly from previous phases consists of the pile forces and their positions, the layer boundary correcting forces, and the soil properties. Further input read from cards determines at what points the stresses are to be computed, and what form the output is to take.

Output may take any number of the three forms mentioned, samples of which are given in fig. 7.1.

POSITION			DIRECT STRESSES			SHEAR STRESSES		
X	Y	Z	XX	YY	ZZ	YZ	ZX	XY
1.0	1.0	10.0	-0.4164E 01	-0.4775E 00	-0.3641E 02	-0.8063E 01	-0.1935E 02	-0.4073E 01
1.0	1.0	20.0	-0.3208E 02	-0.2108E 02	-0.1035E 03	-0.5290E 02	-0.6625E 02	-0.3204E 02
1.0	10.0	10.0	-0.3538E 00	-0.2340E 01	-0.3616E 00	-0.3918E-01	-0.3947E 01	-0.3427E 01
1.0	10.0	20.0	-0.2416E 00	-0.2458E 01	-0.6662E 00	-0.2534E 00	-0.2515E 01	-0.2163E 01

POSITION			PRINCIPAL STRESSES AND THEIR DIRECTIONS											
X	Y	Z	P1	L1	M1	N1	P2	L2	M2	N2	P3	L3	M3	N3
1.0	1.0	10.0	-0.4722E 02	0.42	0.19	0.89	0.4918E 01	0.91	-0.07	-0.41	0.1248E 01	-0.02	0.98	-0.20
1.0	1.0	20.0	-0.1689E 03	0.47	0.39	0.79	0.7513E 01	0.88	-0.12	-0.46	0.4739E 01	-0.08	0.91	-0.40
1.0	10.0	10.0	-0.6161E 01	0.66	0.60	0.45	-0.1413E 01	-0.19	0.72	-0.67	0.4519E 01	0.73	-0.36	-0.58
1.0	10.0	20.0	-0.4525E 01	0.60	0.68	0.43	0.2520E 01	0.76	-0.30	-0.58	-0.1362E 01	-0.26	0.67	-0.69

POSITION			DIFFERENCE BETWEEN PRINCIPAL STRESSES AND DIRECTION OF MAX. P.S.					
X	Y	Z	XY PLANE		YZ PLANE		ZX PLANE	
1.0	1.0	10.0	0.8941E 01	-32.8	0.3939E 02	-12.1	0.5037E 02	-25.1
1.0	1.0	20.0	0.6501E 02	-40.1	0.1341E 03	-26.0	0.1505E 03	-30.8
1.0	10.0	10.0	0.7136E 01	-36.9	0.1980E 01	-1.1	0.7895E 01	-44.9
1.0	10.0	20.0	0.4860E 01	-31.4	0.1862E 01	-7.9	0.5048E 01	-42.5

FIG. 7.1 TYPICAL STRESSES OUTPUT.

CHAPTER EIGHT

PHOTOELASTIC INVESTIGATION OF STRESSES IN A TWO-LAYER ELASTIC SOIL

8.1 SUMMARY

Three dimensional photoelasticity has been used in an attempt to find experimentally the stresses in a two-layer elastic medium due to a laterally-loaded pile. The results obtained were not of a sufficient standard for any conclusions to be reached regarding the theory of Chapters 6 and 7, and time did not permit further tests to be carried out. However, the experimental procedure is of some interest.

8.2 INTRODUCTION

The aim of this experiment was to verify the theory of chapters 6 and 7 by studying stress distributions. It should be noted that the aim was not to predict the behaviour of a real soil: it was an elastic stress distribution which was required. A method which enables a whole two or three dimensional stress field to be examined is photoelasticity.

8.3 PHOTOELASTICITY

The theory and practice of photoelasticity are well documented. A short treatment of the subject is given by Hendry[45].

Briefly, the method allows the determination of stress information from the optical behaviour of transparent materials. A stressed planar model is examined in a polariscope, which in its simplest form consists of two sheets of polaroid with their axes crossed, between which is placed the model, and through which light is shone.

If white light is used, interference effects cause bands of colour to show in the model, monochromatic light giving dark bands. These interference fringes, called isochromatics, are loci of equal difference between principal stresses $(p_1 - p_2)$, and may each be assigned an order n. Values of $(p_1 - p_2)$ may be found anywhere on an isochromatic from the equation

$$p_1 - p_2 = \frac{fn}{t},$$

where f is a constant which depends on the model material and the wavelength of the light, and t is the model thickness. Away from isochromatics, $(p_1 - p_2)$ can be found using the Tardy compensation technique[45].

Another item of information can be gained from photoelasticity: dark lines or areas called isoclinics show where the directions of the principal stresses are the same as the directions of polarisation of the polarising filters. Rotation of the filters will give a set of isoclinics and will enable the principal stress directions at any point to be found.

Photoelastic analysis is not confined to two-dimensional structures because of what is termed the frozen stress effect. This occurs because certain plastics have a molecular structure which may be considered to consist of two phases, the rigidity of one being temperature-dependent. If such a plastic is stressed while above a critical temperature and cooled under load, strains are "frozen" into the model. Fringes which represent an elastic stress distribution can still be observed after removal of the load. Further, the model can be cut into pieces without disturbing the fringe pattern. The load required to produce a given set of fringes is less than would be required below the critical temperature. Thus a three-dimensional model can be stress frozen and then cut into slices for examination in the polariscope.

The best photoelastic materials presently available are epoxy resins, and that most readily obtainable in New Zealand is Araldite which is manufactured by the Ciba Co. Most forms of Araldite are supplied as liquids which require a liquid or powder curing agent. Different Araldite mixes have different chemical and physical properties.

8.4 PRODUCTION OF A CASTING

A paper by Leven[46] proved of great assistance in many matters raised in this section.

8.4.1 Selection of Araldite mix

The main factor influencing the choice of mix was that a relatively thick casting was required. Associated with the curing of Araldite is an exothermic reaction, and successful curing of a thick block cannot be achieved if heat is generated too quickly. High exotherm will cause fast and uneven curing, and because shrinkage occurs during curing, this will introduce casting stresses.

It is desirable also that curing be carried out at elevated temperatures because the decreased viscosity of the mix means that air bubbles introduced during mixing can readily escape.

The above considerations made it inevitable that the curing agent be an acid anhydride because these provide low curing rates, and require elevated temperatures. Phthalic anhydride, sold as hardener HT901, was used.

The curing and final (both mechanical and photo-elastic) properties, while depending greatly on the amount and type of curing agent, are virtually independent of the basic epoxy. That which was used is sold in New Zealand as Araldite F, which is believed to correspond to either Araldite 6010 or 6020 as sold in the United States.

The final properties of a casting depend on the mix proportions and these are discussed in a later section.

8.4.2 Casting Procedure

Casting procedure was basically as described by

Leven.

The oven used has an interior space about 3ft x 3ft x 4ft, and may follow the temperature profile set by a slowly rotating cam. Temperature checks were made by viewing through the windows a thermometer suspended inside the oven. The oven is shown in fig. 8.1.

The oven was set to 92°C and used to preheat the mould. The powdered hardener was heated to about 120°C, when it melted, and then it was mixed thoroughly with the heated Araldite resin. When the temperature of the mixture had dropped to 95-100°C, it was poured into the mould.

Initial gelation of the casting took about one day at 92°C, after which the temperature was slowly raised to 100°C and then slowly lowered to room temperature. The rate of temperature rise must be very slow to produce gradual polymerisation, and the rate of temperature drop must be even slower to prevent the freezing in of thermal stresses. A temperature change rate of just over 1°C/hour was employed, this being in line with that recommended by Leven for the size of casting used.

The temperature was then increased to 150°C, held for 6-7 days, and slowly lowered to room temperature.

8.4.3 Preparation of Mould

The proper preparation of the mould is an important feature if successful castings are to be obtained. Easy parting is essential if casting stresses are to be minimised

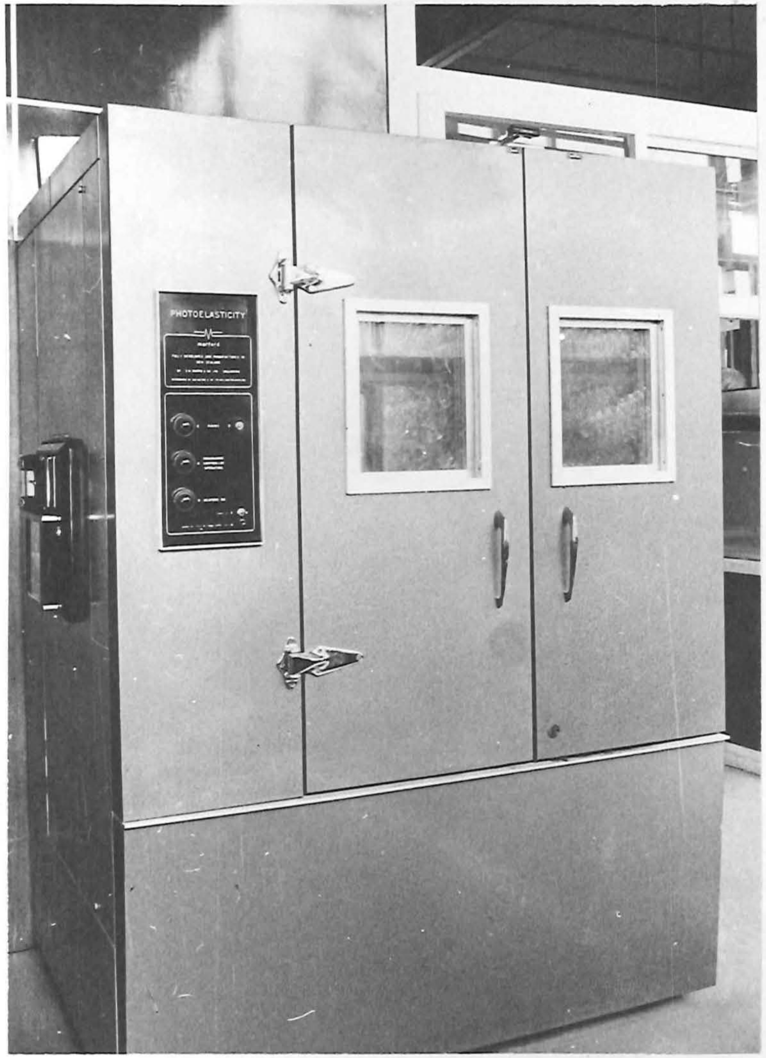


FIG. 8.1 PHOTOELASTICITY OVEN

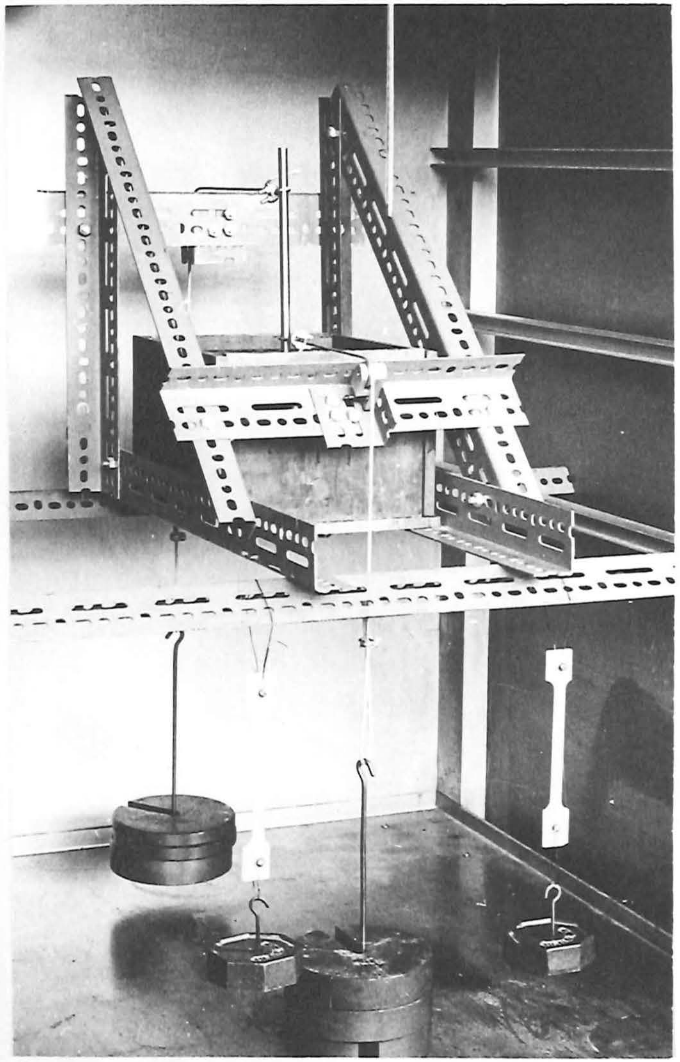


FIG. 8.2 MODEL IN OVEN

and Araldite, which is often used as a glue, provides difficulties in this respect.

Initially, moulds were made of aluminium and a silicone grease solution, Ciba release agent QZ13, was used as a parting compound. This proved unsuccessful. The use of a steel mould provided some improvement but still gave unsatisfactory castings.

Eventually, two successful methods were developed. The first involved the application of a silicone lacquer, Dow Corning "Pan Glaze", before the use of the QZ13. The second used, instead of the Pan Glaze, a silicone rubber as a first mould casting. Any of the silicone rubbers supplied by Dow Corning should be successful: that used was "Silastic" 850 RTV.

Each of these methods has an advantage over the other. The use of Pan Glaze gives a better casting finish. The thin lacquer coating is applied to a machined mould surface and a smooth finish to the casting is obtained. An Araldite casting can be machined, but it will usually be preferable to dispense with this machining. Silicone rubber does not provide such a smooth finish as the coating is more viscous when applied: it is necessarily thicker and subject to changes in thickness and perhaps an undulating surface. However, it has the advantage that the coating is clearly visible and thus easily checked. The same release coating can be reused with confidence.

For this application a smooth finish was not required: silicone rubber was used.

Initially a mould of square section was used, but it was decided that a circular section would minimise casting stresses: the final mould was simply a large glass beaker.

8.4.4 Variation of Araldite Elastic Properties

It was necessary for this test to have two layers of Araldite with different elastic properties.

Initially, it was attempted to produce a more flexible epoxy by the addition to the mix of Araldite 508. The resultant castings were indeed more flexible but were found to be markedly subject to creep. Accordingly, the change in properties was produced simply by varying the amount of hardener used.

Two layers with different properties were separately cast and their faces machined smooth using a carbide tipped flycutter. The two were then glued together using the stronger of the two mixes subjected to the same temperature cycle.

8.5 DESCRIPTION OF MODEL

8.5.1 The Soil

Obviously a semi-infinite continuum cannot be reproduced, and so it was necessary to decide what size block should be used so that boundary effects did not significantly affect the stresses surrounding the pile. A cylindrical block, about 7 in. in diameter and about 7 in. deep, was used, with a pile of length $3\frac{1}{2}$in. The layer boundary was at 1 in. depth. Theoretical analysis of soil stresses showed that they were very small at the boundaries.

The casting was placed inside a square rigid steel box and so some filling was required. This filling, together with the steel box, should as nearly as possible represent the remainder of the half-space, so it should be more flexible than Araldite. It is necessary to have a flexible surround for another reason: the coefficient of thermal expansion of Araldite is higher than that of steel, so that stresses will be set up unless the Araldite is allowed to freely expand. Some sort of rubber is indicated, and the filling used was a silicone rubber, because of its stability at high temperature. The type first tried, Dow Corning Silastic 850 RTV, was found to revert to a liquid state under the conditions of confinement, pressure, and temperature encountered. That finally used was Silastic E. To save cost, the corners of the mould were filled with pre-cast Araldite, leaving a thickness of about $\frac{3}{4}$in. of silicone rubber all round.

The free surface of the block was machined flat.

8.5.2 The Pile

It was not possible to cast the model pile into the soil, as very high casting stresses would result from shrinkage during curing. A hole was carefully drilled into the casting and a high strength steel rod placed as the model pile. Although an initial close fit was achieved, it would have been a little looser at temperature because of the different thermal expansions.

The pile size was determined from the maximum moment it should sustain without yielding. This was $\frac{3}{8}$in. diameter which provided a necessarily stiff pile.

8.5.3 The loading system

It was necessary for measurable stresses to be set up at the layer boundary without soil failure being caused near the surface. The loading applied was thus designed to provide a shear force and a reverse moment at the surface, this system causing displacements of one sense and of comparable order down the length of the pile.

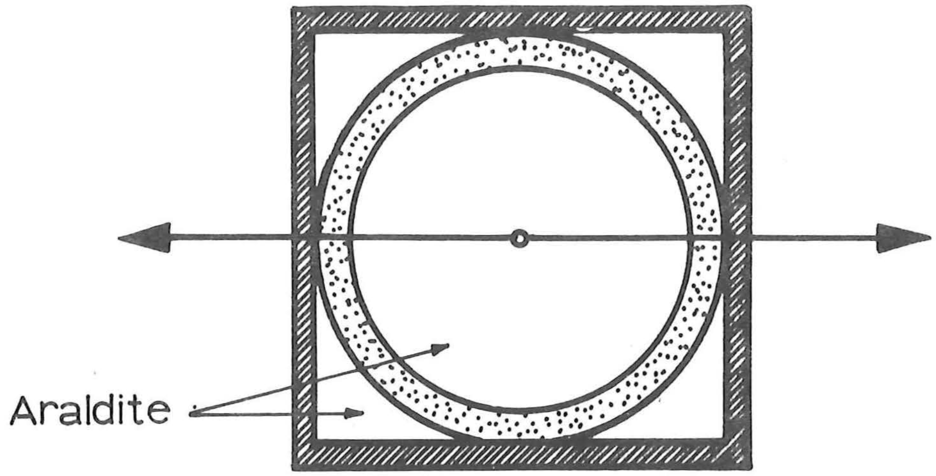
Loading was by means of weights hung from pulleys.

Fig. 8.3 is a schematic diagram of the whole model system, roughly to scale, and fig. 8.2 shows the test setup in the oven.

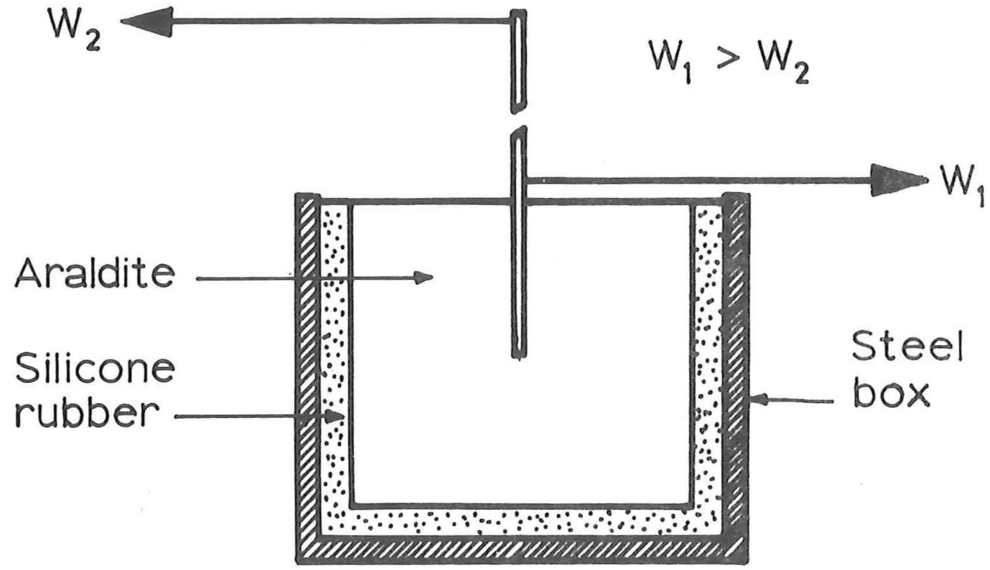
8.6 DETERMINATION OF ARALDITE PROPERTIES

8.6.1 Note concerning critical temperatures

The critical temperature of an Araldite casting depends, like other properties, on the proportion of curing agent in the mix. This fact may at first seem to



Plan



Elevation.

FIG. 8.3 - DIAGRAM OF MODEL.

indicate a problem when a model is made from two different formulations: as the casting is cooled, one layer will change from its flexible to its rigid state while the other remains flexible.

However, this does not imply that a reorganisation of strains occurs. Although the rigid layer now actually has a much higher stiffness (a factor of about 100 is involved) it is still acting as if it were above the critical temperature in that the strain distribution remains unchanged. The strains in the other layer will also therefore remain unchanged and will eventually be frozen in.

It follows that the only properties of interest are those existing at the critical temperature. Thus, for example, Young's modulus can be determined by measuring what strains are frozen into a test piece: no measurements of what actually occurs during the temperature cycle are necessary.

8.6.2 Measurement of Elastic Properties

Measurements of the shear modulus G and Poisson's ratio μ were made on specimens loaded in tension. Slices were bandsawn from the casting and machined on both faces using a carbide-tipped flycutter. Specimens with a shape typical of tensile tests were prepared from these using a high speed profile router with a carbide tipped bit.

Fine lines were scribed across the test specimens and the distances between them measured with a travelling microscope. Width and depth dimensions were obtained with a micrometer. The specimens were hung inside the oven and loaded at the same time as the model. They were, of course, subjected to the identical temperature cycle.

After unloading, length, width and depth measurements were retaken and G and μ calculated.

8.6.3 Measurement of Photoelastic Properties

The same specimens were used to find the photoelastic constant f. For a bar in tension, p_2 is zero and p_1 is easily calculated. Thus f can be found from

$$f = \frac{p_1 t}{n} \quad .$$

8.7 EXPERIMENTAL DETAILS

8.7.1 Mix formulation

Two models were made, one with constant properties and the other with two layers. The parts by weight are given in table 8.1

	Araldite F	Hardener HT901
Constant properties model and top layer of two-layer model	100	60
Bottom layer of two-layer model	100	40

Table 8.1 - Composition of Araldite mixes

8.7.2 Stress freezing temperature cycle

The temperature of the unstressed model was slowly raised to about 150°C, above the critical temperature of both layers. The load was then applied in several increments over 3-4 hours and the temperature slowly lowered.

8.7.3 Photoelastic Equipment

The polariscope used has a wide field, with polarising plates of diameter 18 in., and quarter-wave plates to remove the isoclinics if desired. A diffuse light source can provide either white or, with the help of a filter, monochromatic light. It was set up in a darkened room.

Photography was used as a means of recording isochromatic patterns. A plate camera with a 21 in focal length lens was used with monochromatic light.

Fig. 8.4 shows both the polariscope and the camera.

8.7.4 Preparation of slices

Slices for examination were prepared as previously mentioned, being carefully bandsawn from the model and machined smooth with a flycutter. Before photography, the slices were wet with tricresyl phosphate, which has approximately the same refractive index as Araldite, to remove any machining marks.

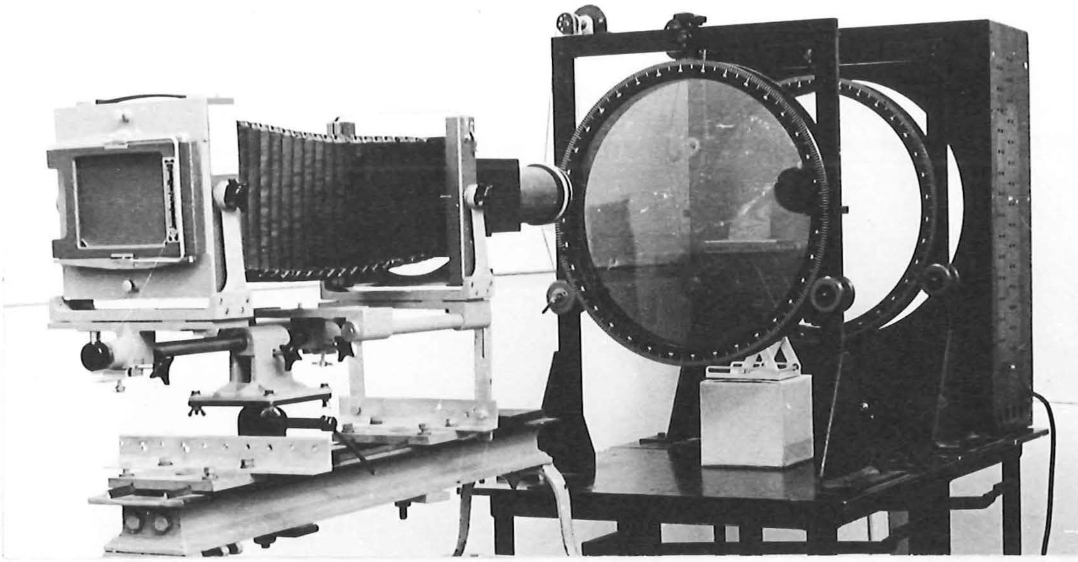


FIG. 8.4
CAMERA AND POLARISCOPE

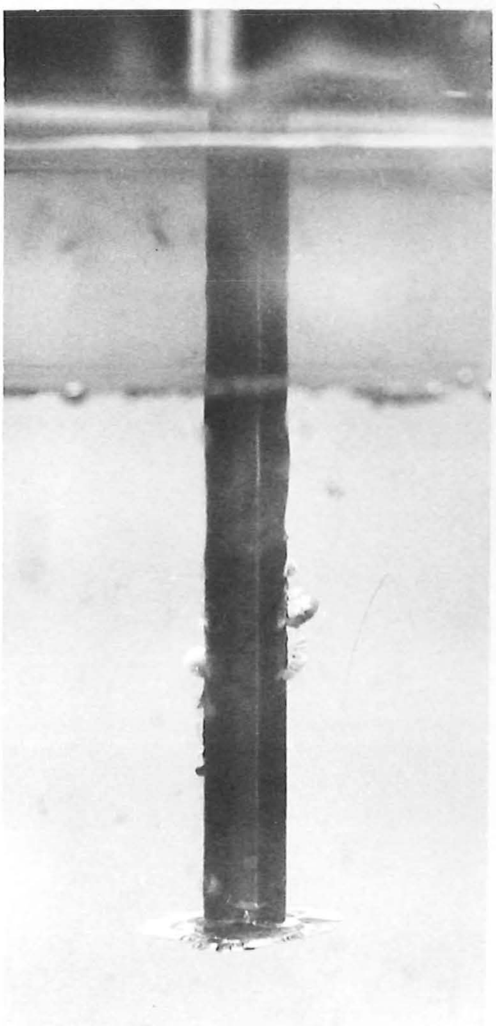


FIG. 8.5
MODEL PILE, SHOWING FAILURE SURFACES

8.8 MEASUREMENTS AND OBSERVATIONS

8.8.1 Araldite Properties

Two test specimens were made from each mix and gave properties as set out in table 8.2.

Curing Agent Content	40%			60%		
Specimen No.	1	2	Value Used	1	2	Value Used
E (psi)	1350	1405		4100	3710	
μ	.500	.547	.500	.451	.508	.500
f (psi/fringe/in)	1.84	1.84	1.84	2.04	2.05	2.05
G (psi)			460			1300

Table 8.2 - Araldite Properties

It can be seen that there are quite large variations in the values of E and μ. These variations are within the ranges of reading errors. As all final readings involved the small difference between two nearly equal quantities, values are to an accuracy of only ± 15%. A value for Poisson's ratio of .500 may be used with confidence as this is the expected value (Durelli and Riley[47]). The values of E show that it would have been preferable to use more test specimens. However, it is unlikely that the average values of E will be more than a few per cent in error, and the stress distributions will not be greatly affected by this error. G was calculated from E using a value for μ of .5.

The values of f were easily determined with accuracy.

A comparison with typical properties given by Leven for Araldite 6020 shows that all properties correspond with those given for mixes of lower curing agent content.

8.8.2 Other Measurements

The constant properties model was loaded with 120.63lb at a height of .375 in. above the surface and 25.63lb. in the opposite direction at 11.80in. so that

$$V = 95\text{lb}$$

and $$M = -257\text{lb.in.}$$

The two layer model was loaded with 92.63lb at .46in and 15.63lb at 9.10in so that

$$V = 77\text{lb}$$

and $$M = -99.6\text{lb. in}$$

The depth of the layer boundary was 1.0in, the top layer being the stiff one.

The pile length was 3.50in, the width .375in, and the stiffness EI 2.91×10^4 lb.in^2.

8.8.3 Cracking of Araldite

It was noticed that the two-layer model had been slightly overloaded, causing some failure surfaces to develop. These can be seen in fig. 9.5. The photograph was taken with the block immersed in tricresyl phosphate and the bubbles at the layer boundary are not in the Araldite. They occurred because the top layer had a slightly larger diameter than the bottom layer, causing a step.

8.9 COMPARISONS OF EXPERIMENT WITH THEORY

8.9.1 Basis for comparison

For a two-dimensional model, photoelasticity alone does not provide enough information to completely define the stress state at a point. In a three-dimensional case, however, the analysis of slices cut in three planes provides a complete statement of the stress state.

Consider a slice cut perpendicular to the x_1 axis. Isochromatics give

$$2\tau_1 = (\sigma_{22} - \sigma_{33})^2 + 4\,\sigma_{23}^2 ,$$

where τ is the maximum shear stress in the plane of the slice. Isoclinics give

$$\theta_1 = \tfrac{1}{2}\tan^{-1}\frac{2\,\sigma_{23}}{\sigma_{22} - \sigma_{33}}$$

so that $4\,\sigma_{23}^2 = (\sigma_{22} - \sigma_{33})^2 \tan^2 2\theta_1$

and $$\sigma_{22} - \sigma_{33} = \frac{2\tau_1}{\sec 2\theta_1}$$

Similar equations may be derived for planes perpendicular to the x_2 and x_3 axes, enabling σ_{11}, σ_{22}, σ_{33}, and hence σ_{23}, σ_{31}, σ_{12} to be found. Thus if slices are cut perpendicular to the three axis directions and the isochromatic and isoclinic patterns match with theory, the stress systems are identical.

The program STRESS, described in the previous chapter, has therefore been written so that output can be requested in a form which enables a direct comparison with photoelastic patterns to be made. This is the third form shown in fig. 7.1.

8.9.2 Comparisons

Figs 8.6-8.8 compare theoretical and obtained isochromatic patterns. Fig. 8.6 shows reasonable agreement but figs 8.7 and 8.8 show the experimental patterns to be not good enough for proper comparisons to be made. Fig. 8.6 is the top horizontal slice in the two-layer model. Figs 8.7 and 8.8 are vertical slices in the place of loading, with one face through the centre of the pile. Fig. 8.7 is from the constant properties model and fig. 8.8 from the two-layer model. Because of the poor results, other slices are not compared, nor are isoclinic patterns given.

8.10 DISCUSSION

A number of procedural faults can be ascertained from figs 8.6 - 8.8 and are listed below.

8.10.1 Cracking of Araldite

The previously mentioned cracking can be seen at the pile tip in fig. 8.8, where a stress concentration occurs. However, the cracking is not extensive and should not have greatly affected the stresses near the layer boundary.

8.10.2 Casting faults

Irregularities and "flow lines" are apparent in all of figs 8.6 - 8.8. There is an obvious irregularity in the stress pattern in fig. 8.7, and vertical irregularities in figs. 8.7 and 8.8. Flow lines are evident in all cases.

Two possible causes are advanced for these effects:

(1) It is possible that the second (silicone grease) parting compound layer was not entirely dry. The solvent could thus have caused inclusions.

(2) The mix may have been allowed to cool to too low a temperature before being placed in the oven. The cooling to about 95°C was allowed so as to eliminate the possibility of a highly exothermic reaction developing, but the mix should perhaps have been poured at the solution temperature of about 120°C.

8.10.3 Boundary Stresses

Boundary stresses have obviously been introduced and two causes can be identified.

(1) Boundary irregularities have introduced stresses, particularly the step at the layer boundary.

(2) The silicone rubber used was almost definitely too stiff. This will have restricted the thermal expansion

FIG. 8.6 THEORETICAL AND EXPERIMENTAL ISOCHROMATIC PATTERNS (1)

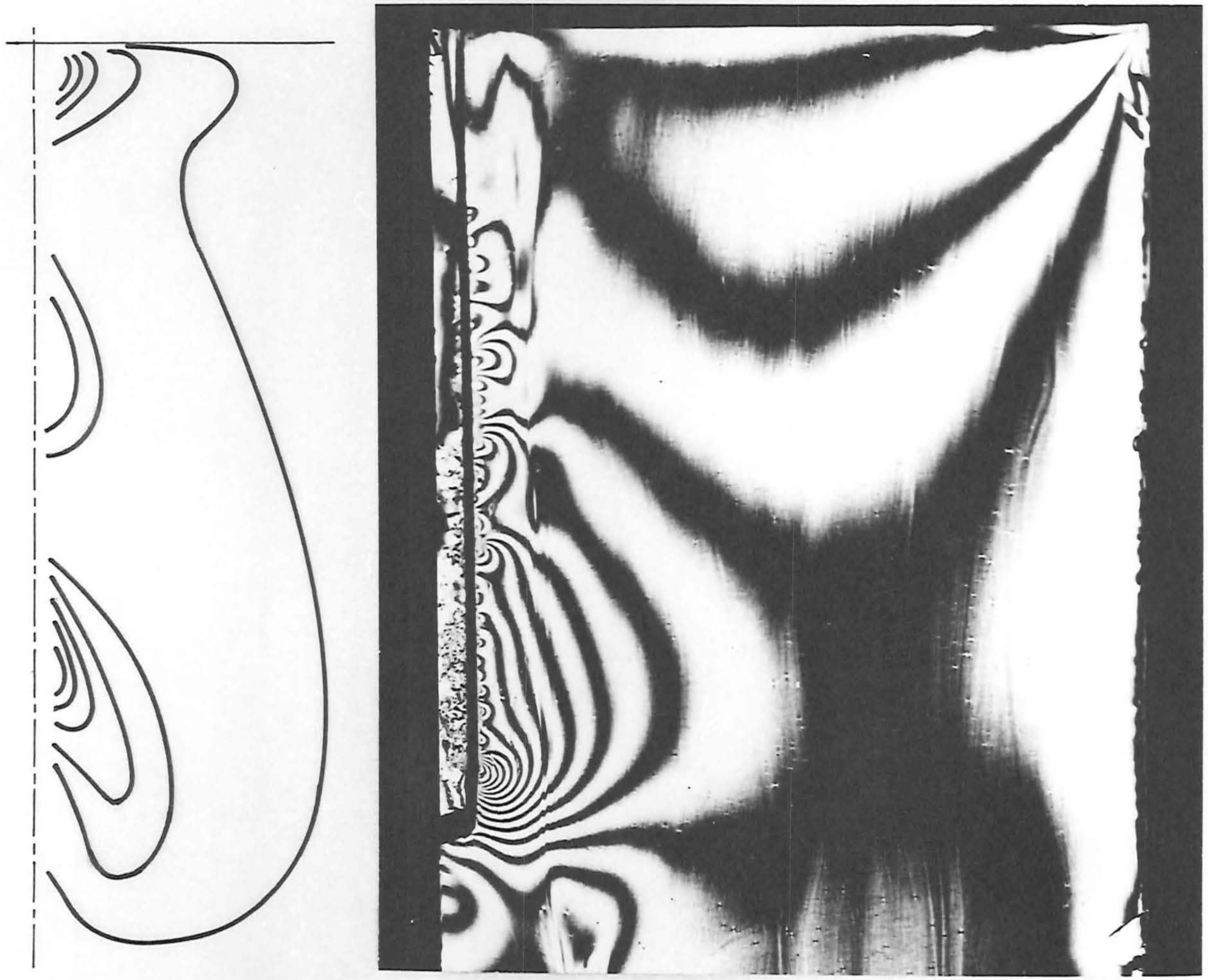


FIG. 8.7 ISOCHROMATIC PATTERNS(2)

FIG. 8.8 ISOCHROMATIC PATTERNS (3)

of the Araldite and upset stress patterns throughout the model. A silicone foam rubber would probably be a suitably flexible replacement for the solid rubber used. This is manufactured in the United States, but it is not stocked in New Zealand.

8.10.4 Drilled Holes

The small stress concentrations obvious along the pile face in fig. 8.6 are evidently due to the inclusion of small Araldite particles in the hole drilled for the pile. The hole was cleaned out with a certain amount of care, so extreme thoroughness must obviously be used.

The fact that the pile was placed in a drilled hole rather than cast in place has produced an interesting effect. It was found that in general stresses were not symmetrical about the horizontal axis perpendicular to the loading direction, being lower on the tension side. This accounts for the absence of the expected stress concentration at the top of the pile in fig. 8.6, where the pile is tending to part from the Araldite. This effect is contrary to elastic theory, and raises the question as to what might occur in practice. If the pressure of overburden is high, there should be no problem, but near the surface some allowance may have to be made. This is a problem which can only be resolved by pile tests.

8.10.5 Pile thickness

There will be some inadequacy of the theory as far as stresses close to the pile are concerned because the theoretical pile is assumed to have width but no thickness. This will cause some re-arrangement of stresses very close to the pile, but should not be significant in practice.

8.11 CONCLUSION

Although a considerable amount of time was spent on this experiment, the aim has not been met, because the results are not of a sufficient standard.

The experimental technique could probably be improved sufficiently to give meaningful results, but time did not permit a repetition of the test. The total time for casting, machining, glueing and testing is over one month.

The experiment has been reported for interest and because the report should help any future three-dimensional photoelastic work.

CHAPTER NINE

FULL-SCALE PILE TEST

9.1 SUMMARY

A full scale test on a laterally loaded free head pile is described.

The test was carried out in conjunction with the Lyttelton Harbour Board, whose main aim was to predict fixed-head pile behaviour. It has been reported with this aim in mind but as far as this thesis is concerned there are several other points of strong interest.

The test procedure was very successful.

The system behaviour proved to be very close to elastic.

Some limited soils information was available: this proved inadequate for accurate prediction but nevertheless indicated that the determination of soil constants should provide no real difficulties once further research has been carried out. The test emphasised the need for such further research.

9.2 INTRODUCTION

At the time of writing, the Lyttelton Harbour Board is planning to build a new roll-on cargo berth.

Alternative mooring jetty designs feature

(1) A conventional "rigid" jetty structure with raker piles and a separate fender-pile system, and

(2) An interesting scheme in which the impact energy of a ship is absorbed by the deflection of the whole jetty, which is supported by vertical piles alone.

In the latter case, the jetty would be built in sections keyed together so that initial displacement was taken by one section and further displacement would involve adjacent sections as well.

To properly evaluate the feasibility of the second scheme it was decided to carry out a lateral load pile test.

The test was financed by the Harbour Board, and was carried out under the supervision of Mr. H. Mills, in consultation with the writer. Most of the instrumentation was provided by the University of Canterbury.

Some soils data was available in the form of

(1) the pile driving record

(2) some triaxial tests performed on samples from a nearby site, and

(3) a test bore record, also from a nearby site.

For the Lyttelton Harbour Board, the aim of the test was to predict the behaviour of a fixed-head pile. It was decided that this could be done with reasonable accuracy, using the computer program developed during this research project, if a free-head pile were tested. The justification for this decision is discussed in a later section.

As far as this thesis is concerned, the main interest was in whether the predicted soil properties were close to correct. If so, the method of analysis would appear to be realistic and the determination of soil elastic properties easy.

9.3 DESCRIPTION OF TEST

9.3.1 General

The pile used was a 110ft long 2 ft diameter concrete filled steel tube. It was driven to a depth of 60 ft close to the pile supported breastwork at the site of the proposed jetty. The location was clear of existing piles, with none in line with the direction of pull and with a clearance of several feet on either side.

Fig. 9.1 shows the pile in place.

9.3.2 Load Application

The horizontal load was applied through a wire rope from the winch of a Caterpillar D8 tractor which was situated about 60 ft away on the breastwork; far enough that no load would be transmitted through the soil to the test pile. A load cycle took about 30 seconds: 10 seconds each for loading, holding and unloading. This was intended to simulate the short-term loading of a ship striking the

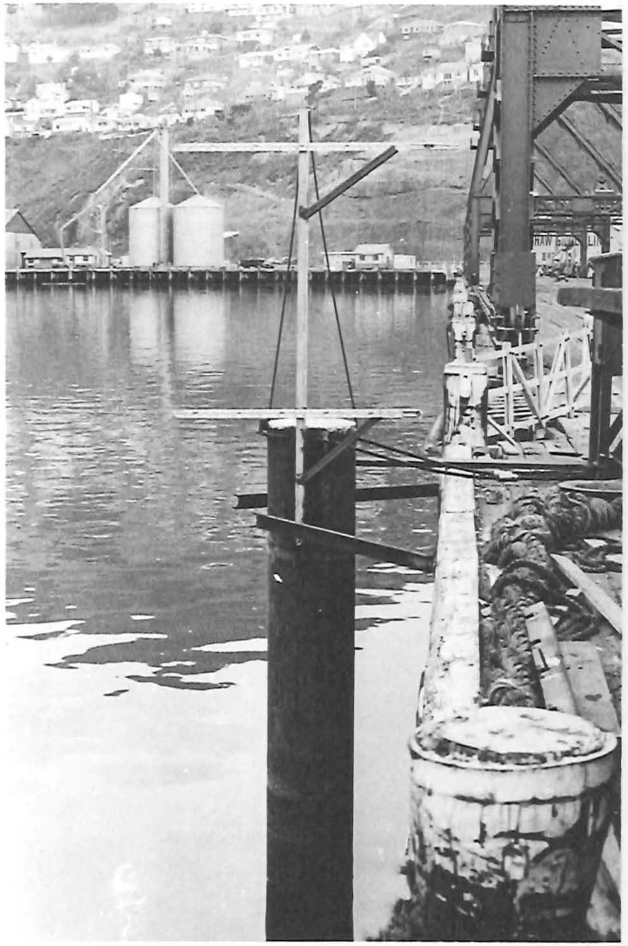


FIG. 9.1 TEST PILE IN PLACE

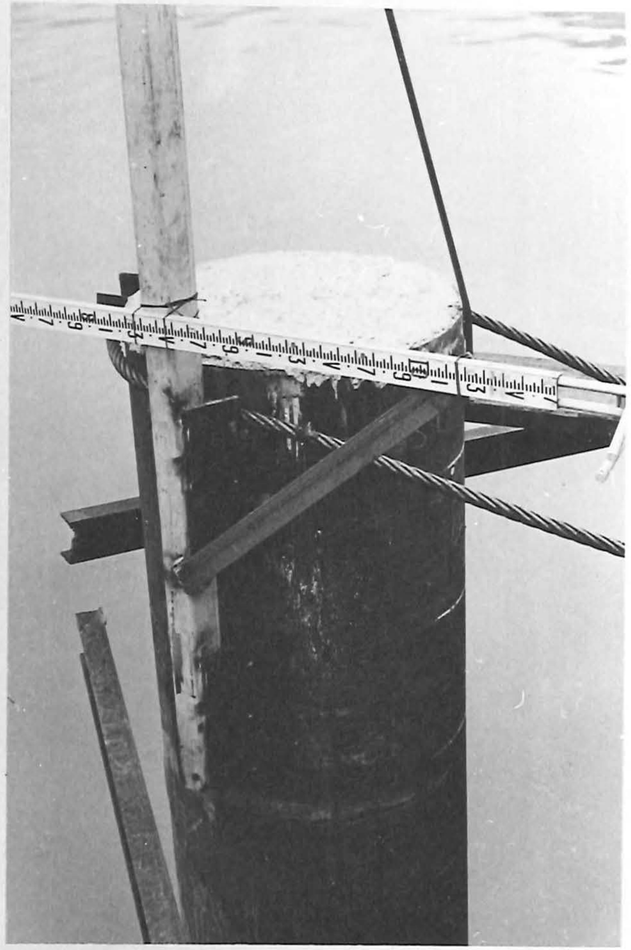


FIG. 9.2 CLOSE-UP OF PILE HEAD

FIG. 9.3 VIEW OF TEST

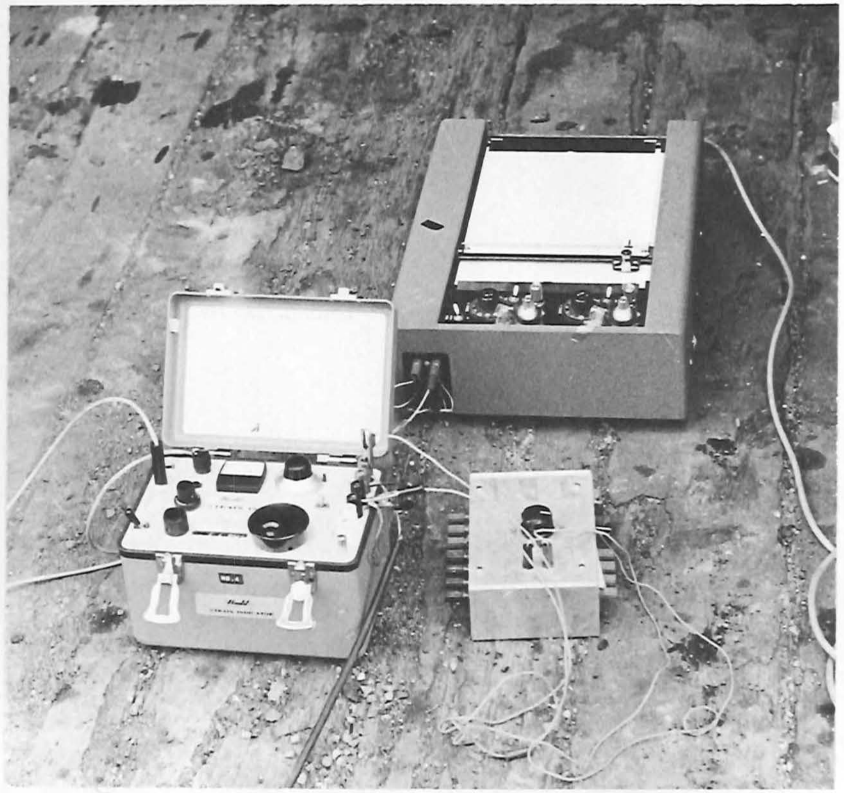


FIG. 9.4 INSTRUMENTATION

jetty. After initial trials, two runs were made, each of 7 cycles to increasing loads until a maximum displacement of about two feet was obtained. Each run took about ten minutes to complete, and there was a 15 minute pause between runs. Control of loading rate was good and the winch brake was used to hold the load steady at a value just below the peak.

9.3.3 Deflection measurement

Both the displacement and the rotation at the pile top were measured. Two horizontal scales six feet apart were set up and read from separate fixed theodolites placed about 60 feet away perpendicular to the direction of pull. Level staffs with divisions of 0.01 foot were used as scales and could be read quickly and accurately. The staffs may be seen in figs 9.1 and 9.2, and the theodolites set up in fig. 9.3.

9.3.4 Load Measurement

The load was measured using a loading link which was a strain gauged steel bar of about two inches diameter. Output from a Budd P350 strain bridge was recorded using one axis of a Riken-Denshi XY plotter. The link had been recently calibrated, establishing a reliable load-strain reading relationship. The strain reading-recorder reading relationship was measured using a simple calibration device built by Mr P. G. Johnstone. This provided a variable known resistance change (equivalent to a known strain), and the corresponding movement of the recorder pen could be read. Three calibration runs were made during the test to check for drift. The load measuring apparatus is shown in fig. 9.4.

This method of measuring load proved most satisfactory.

9.4 EXPERIMENTAL READINGS

9.4.1 Pile Measurements

Pile measurements were as follows:

Total length = 110 feet

Embedded length = 58 feet

Outside diameter = $24\frac{1}{2}$ inches

Steel thickness = 5/16 inches

Concrete crushing strength on day of test

= 6340
6240
6320
6210

Ave 6280 psi

whence [48] Young's modulus for concrete

$$E_c = 4.56 \times 10^6 \text{ psi}$$

9.4.2 Deflection Readings

Table 9.1 shows the deflection readings on both scales. When the load was applied it reached a peak and then fell slightly, as the brake was applied, to a constant value. The deflection readings correspond to the

RUN AND CYCLE		TOP SCALE			BOTTOM SCALE	
		MIN	MAX		MIN	MAX
1.1		4.00 3.99	3.52		8.00 7.99	7.57
1.2		3.99 3.99	3.40		7.99 7.99	7.47
1.3		3.99 3.98	3.02		7.99 7.98	7.12
1.4		3.99 3.96	2.49		7.99 7.97	6.64
1.5		3.97 3.95	2.34		7.97 7.95	6.51
1.6		3.96 3.93	2.09		7.96 7.94	6.29
1.7		3.94 3.89	1.77		7.95 7.90	5.99
2.1		3.92 3.93	3.55		7.93 7.93	7.59
2.2		3.93 3.93	3.16		7.93 7.93	7.24
2.3		3.93 3.93	2.92		7.93 7.94	7.03
2.4		3.94 3.96	2.51		7.94 7.96	6.66
2.5		3.96 3.90	2.16		7.96 7.90	6.34
2.6		3.91 3.87	1.87		7.91 7.88	6.08
2.7		3.89 3.84	1.67		7.89 7.85	5.90
After 12 minutes		3.89			7.89	

On cycle 2.4, the load was released more rapidly

Table 9.1 - Deflection readings

held value rather than the peak.

The deflection was taken to be the difference between the maximum reading and the zero reading for that cycle, rather than the initial zero reading.

As the scales were not exactly parallel to the direction of pull a correction factor of

$$1 / \cos(\sin^{-1} 6/35) = 1.015$$

was applied to the measured deflections. Corrected displacements and slope values are given in table 9.2.

9.4.3 Load Readings

It was found that 1 mm. of pen travel on the recorder represented a load of 56.25 lb. Load values for each cycle are given in table 9.2.

Fig. 9.5 shows load-displacement and load-slope curves plotted from table 9.2. These curves are discussed later.

9.4.4 Soils Data

The pile driving record is given in table 9.3 and it may be deduced from this that there are effectively three soil layers with boundaries at depths of 19 and 35 feet. A triaxial test result for the material of the middle layer was also available.

9.5 PROCESSED RESULTS

9.5.1 Introduction

As has already been stated, the aim of the test was to predict the behaviour of a fixed-head pile. It is

RUN AND CYCLE	LOAD (lb)	DISPLACEMENT (ft)	SLOPE x 1000
1.1	1570	.43	8.3
1.2	1920	.53	11.7
1.3	3150	.88	16.7
1.4	5010	1.36	25.0
1.5	5390	1.47	28.4
1.6	6330	1.69	33.3
1.7	7400	1.97	35.0
2.1	1290	.34	6.7
2.2	2530	.70	13.3
2.3	3340	.91	18.3
2.4	4720	1.31	25.0
2.5	5960	1.61	30.0
2.6	7030	1.83	33.3
2.7	7650	2.00	38.4

Average load/displacement = 3710 lb/ft

Average load/slope = 1940 x 10^3 lb

Table 9.2 - Load and deflection values

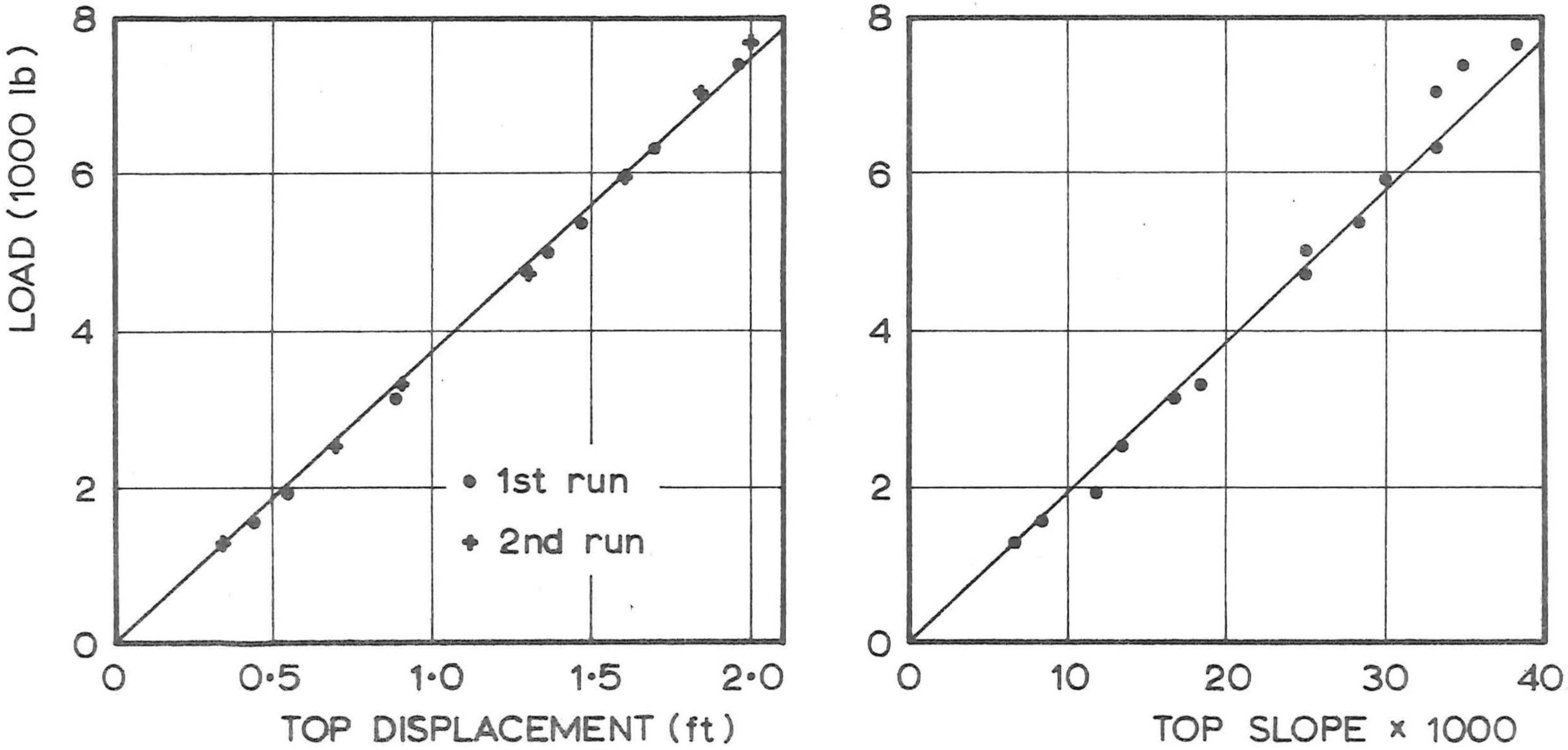


FIG. 9.5 - LOAD-DISPLACEMENT AND LOAD-SLOPE CURVES.

DEPTH (ft)	NO. OF BLOWS	DEPTH (ft)	NO. OF BLOWS	DEPTH (ft)	NO. OF BLOWS
0	2	17	7	34	22
1	2	18	7	35	26
2	2	19	16	36	33
3	2	20	16	37	46
4	2	21	13	38	33
5	2	22	13	39	36
6	2	23	13	40	36
7	3	24	13	41	38
8	3	25	18	42	49
9	3	26	20	43	61
10	3	27	20	44	41
11	4	28	18	45	50
12	4	29	19	46	52
13	3	30	18	47	53
14	3	31	18	48	62
15	3	32	18	49	80
16	4	33	14		

Down to depth 19 ft, the hammer fall was about $4\frac{1}{2}$ ft. After that, it was approximately 8 ft.

Table 9.3 - Pile Driving Record

necessary to outline now the steps which were taken in order to fulfil this aim.

The soil properties were roughly predicted from the pile driving record and the triaxial test and then modified so that the theoretical top displacement and rotation matched the experimental values. The modified values were then used to predict fixed-head pile behaviour.

For such a prediction to be valid, two conditions must be met. They are:

(1) The soil is elastic (so the principle of superposition holds), and

(2) The soil properties are correct.

Condition (1) is in this case close to being met as is shown by the curves in fig. 9.5. Condition (2) requires further discussion.

We may write the following equations for the top displacement Δ and slope θ in terms of the applied shear V and moment M:

$$\Delta = d_v V + d_m M$$

and $$\theta = s_v V + s_m M$$

As $d_m = s_v$ we may write for the fixed head case

$$\Delta = d_v V + s_v M$$

and $$\theta = 0 = s_v V + s_m M$$

The test has given values for the flexibility coefficients d_v and s_v, but the two equations still contain the three unknowns Δ, M, and s_m and so are not soluble.

It thus appears that there is not necessarily a unique set of soil properties providing the correct values for d_v and s_v and it will not in general be sufficient simply to obtain a set of soil properties which give such correct values. In this case, however, the evidence provided by the soils data has allowed condition (2) to be met with confidence.

9.5.2 Initial Soil Properties

It has been mentioned that the pile driving record shows the soil to consist of three layers with boundaries at 19 and 35 feet. That this is reasonable was confirmed by a bore hole record from a nearby site which showed three layers with boundaries at 24 and 44 feet, the top and bottom layers being clay and the middle layer being a sand-clay mixture.

The energy/foot required to drive the pile in each of the layers, which is proportional to the number of blows multiplied by the hammer fall, can be calculated as follows.

Average number of blows, 0-16 ft = 2.7

19-34 ft = 16.6

35-48 ft = 44

Therefore Energy/foot, top layer $\propto$ 2.7 x 4.5 = 12

middle layer $\propto$ 16.6 x 8 = 133

bottom layer $\propto$ 44.0 x 8 = 352

As a rough estimate, the shear modulus G was taken to be proportional to this energy.

G for the middle layer was computed as follows from the triaxial test.

The submerged density was calculated using

$$\gamma = \frac{(H - 1)\ \gamma w}{1 - wH}$$

where w is the moisture content, H the specific gravity and γ w the density of water.

The confining pressure could then be calculated by multiplying γ by the average depth of the layer, and the appropriate triaxial test was that which used a similar confining pressure.

Expressing the stress strain relationship in the form

$$e_1 = \frac{1}{E} \left[\sigma_1 - \mu(\sigma_2 + \sigma_3) \right]$$

we have, since $\sigma_2 = \sigma_3$ = confining pressure,

$$e_1 = \frac{1}{E} (\sigma_1 - 2\mu\sigma_3)$$

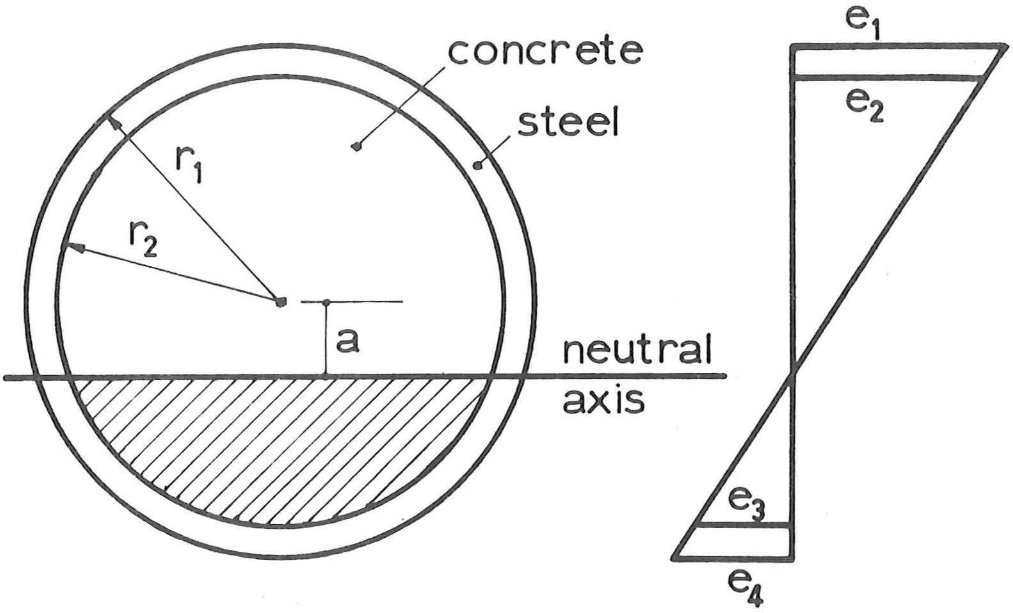
Since the soil is submerged, we may use $\mu = .5$

So $e_1 = \frac{1}{E} (\sigma_1 - \sigma_3)$

and $G = \dfrac{E}{2(1+\mu)}$

$$= \frac{1}{3} \frac{(\sigma_1 - \sigma_3)}{e_1}$$

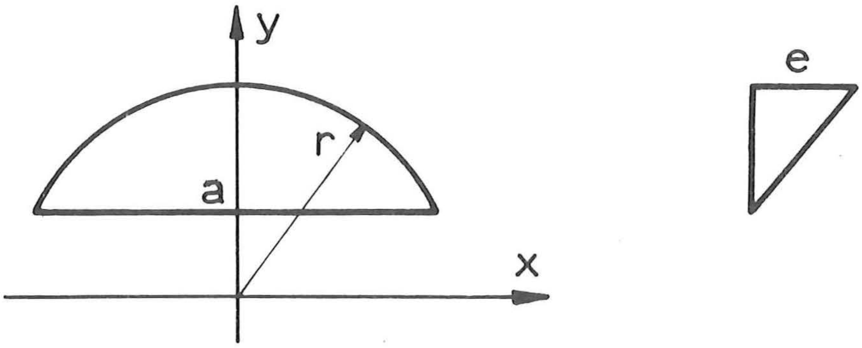
Thus G is equal to $\frac{1}{3}$ of the slope of the



PILE SECTION

STRAIN DIAGRAM

FIG. 9.6(a) - PILE IN BENDING.



FRONT ELEVATION

SIDE ELEVATION

FIG. 9.6(b) - VOLUME TO BE FOUND.

The equation of the circle is

$$x^2 + y^2 = r^2$$

and $$dV = 2x.dy.\frac{y-a}{r-a}.e$$

so $$V(e,r,a) = \frac{2e}{r-a}\int_a^r x\ (y-a)\ dy$$

$$= \frac{e}{6(r-a)}\left[3r^2a(2\sin^{-1}\left(\frac{a}{r}\right)-\pi)+2\sqrt{r^2-a^2}\ (2r^2+a^2)\right]$$

The forces may now be calculated.

$$T_s = E_s\left[V\ (e_1,\ r_1,-a)\ -V\ (e_2,\ r_2,\ -a)\right]$$

$$C_s = E_s\left[V\ (e_4,\ r_1,\ a)\ -V\ (e_3,\ r_2,\ a)\ \right]$$

$$C_c = \quad E_c.\ V\ (e_3,\ r_2,\ a)$$

r_1 and r_2 are known, and all the strains may be expressed in terms of r_1, r_2, and e_1. It is thus possible to select a, and calculate all the forces in terms of e_1.

A short computer program was written to find a by trial and error so that

$$T_s - C_s - C_c = 0.$$

a was found to be 3.36 in.

The bending stiffness of the composite section could then be computed and was found to be 7.2×10^{10} lb in^2. It is interesting to compare this with EI for the steel alone(with the neutral axis at the centre of the section), and with the combined EI if the concrete can take the full tension stress: values obtained are 5.2×10^{10} and 12.5×10^{10} lb in^2 respectively.

It is now necessary to consider how this value may be affected by the ability of the concrete to sustain some tensile stress.

The concrete stress due to shrinkage can be calculated as follows:

Concrete strain / steel strain (e_c/e_s) = axial stiffness of steel / axial stiffness of concrete (EA_s/EA_c) = .086.

Now $e_c + e_s$ = the unrestrained shrinkage strain which will be taken as 200×10^{-6}.

Whence tensile stress in concrete

$$= E_c \, e_c$$

$$= 100 \text{ psi.}$$

As the concrete would be expected to take 600-1000 psi in tension, it is obvious that some tension will be carried.

Before the test was carried out some trial runs were made up to a top displacement of about two feet, the maximum recorded in the test. This means that, using the figures quoted above, cracking at the point of maximum moment would have extended to within about two inches of the neutral axis position. This provides an almost negligible increase in pile stiffness. However, at sections of low moment (near both the top and bottom of the pile) the increase is large.

The stiffness near the pile tip will have negligible effect on the ground level slope and displacement but above ground level the increased stiffness will have significant effect on values at load level. Accordingly it was decided to use a constant value of 7.2×10^{10} lb in^2 below ground level and to assume a linear variation from 7.2 at ground level to 12.5×10^{10} lb in^2 at load level, using hand calculations for that part of the pile above ground level.

9.5.4 Final soil properties and prediction of fixed-head pile behaviour

Two facts became clear after several sets of soil properties had been tried. Firstly, the initial values obtained in section 9.5.2 were somewhat high. Secondly, changing of properties in the middle layer had only small effect, and in the bottom layer almost negligible effect. This second fact confirmed the validity of neglecting the increase in pile stiffness at depth.

The fixing of the layer boundary positions proved to restrict greatly the way in which the top displacement and slope could be varied independently, unless radical changes were made in the ratios between predicted soil properties. The fact that both displacement and slope could be matched without too great a departure from the predicted soil properties gave added confidence that both the soil properties and the pile stiffness were essentially correct.

It was found that reducing G for the top layer to 13.5 psi and leaving middle and bottom layer properties at the predicted values gave a correct top displacement and a top slope which differed from the experimental value by less than two per cent.

The values of G used were thus 13.5, 200, and 500 psi for the top, centre, and bottom layers. The proposed piles are 3/8 in. steel thickness, and they will be fixed 4 feet below test load level. The value of a was found to be 3.0 inches, and the pile stiffness, assuming the concrete to have no tensile strength, is 8.38×10^{10} lb in^2. These values gave a pile top displacement of 1.54 inches/ton and a fixing moment of 970,000 lb in/ton.

9.6 DISCUSSION AND CONCLUSIONS

9.6.1 Test procedure

The methods of load application, load measurement, and deflection measurement all proved most satisfactory.

Many previously reported pile tests have involved the jacking apart of two piles placed together. Because of the closeness of the piles the soil will have been effectively stiffened by interaction effects, and the results would not always accurately predict the behaviour of a pile loaded under different circumstances. The load application technique used here did not suffer from this defect. In addition, it allowed generally very good control of the load.

Load and deflection measurements could both be taken quickly and, as the small scatter of points in fig. 9.5 shows, with accuracy.

The test could have been made more complete in several ways. Separate application of the load at two levels would have been a simple way to allow the prediction of fixed-head pile behaviour without soils information; a test using just the steel tube would have eliminated any doubt as to the pile stiffness; and strain-gauging of an unfilled pile would have allowed a bending moment distribution and hence a full deflected shape to be obtained. However, it is felt that the test as carried out has fulfilled the aims with more than sufficient accuracy.

9.6.2 Elasticity of Behaviour

The elasticity of the system behaviour has proved both interesting and rather pleasing. Fig. 9.5 shows that, in fact, the stiffness increased slightly with load. The values of initial set, obtainable from table 9.1 , indicate virtually elastic behaviour. After a second displacement of about two feet, a set of about $1\frac{3}{4}$ in. was observed, but the pile was returning to its initial position at a relatively fast pace. The accidental more rapid release of load in cycle 2.4 and the corresponding decrease in set led to a strong feeling that a sudden release of load would have allowed a return almost to the initial position. No creep was noticeable while the load was held.

It seems possible that, for clay soils, and short term loading, behaviour may in general be elastic up to a quite high displacement level. In this case a maximum ground level displacement of almost 4in. was achieved with very little sign of plastic behaviour.

9.6.3 Soil Properties

There is no reason to suspect the accuracy of the values finally obtained for the soil properties. It has been mentioned that it was necessary to radically change the predicted properties if a change in slope alone or displacement alone was required. That both were matched without such change implies that approximately correct soil properties were found.

The assumption that G is proportional to the pile driving energy was, of course, intended to provide only a very rough estimate of properties. The change of top layer G from 18 to 13.5 psi is reasonable when viewed in this light.

Although pile top behaviour was not greatly sensitive to middle layer stiffness, the triaxial test prediction was certainly reasonable. This particular triaxial test was not intended for the prediction of G and if a test were carried out for such a prediction some improvements could be made. The most obvious is that the sample should be loaded and unloaded several times, because the slope of the unloading line, which represents elastic behaviour, is much more easily determined than the initial slope of the loading line. Also, it would be preferable to use a sample which had a horizontal orientation within the soil mass, in case the soil is more than a little anisotropic.

Geophysical techniques may prove to be the best method of obtaining soil elastic constants, but the properly planned triaxial test also seems promising. In any case, it appears that further research into methods of determining soil elastic properties has an excellent chance of success. There is an obvious need for such further research.

CHAPTER TEN

CONCLUSION

10.1 REVIEW OF THEORETICAL WORK

10.1.1 Soil as an Elastic Continuum

The basis of this investigation has been the idea that the best initial step towards the solution of laterally loaded pile problems is to treat the soil as elastic and continuous.

The bulk of previous work on the topic has used the Winkler soil model, which may be thought of as a row of unconnected springs, the spring stiffness being called the horizontal modulus of subgrade reaction k_h. However, the elastic continuum has a number of advantages over the Winkler model:

(1) It is a more accurate representation of real soil.

(2) It allows stresses within the soil to be determined, and

(3) The constants used to describe the soil are dependent on only the soil properties, while k_h depends on pile properties and loading as well.

10.1.2 Behaviour of pile in elastic half-space

An analysis has been developed which allows the solution by computer of a laterally loaded pile in an elastic half space, and the solution has been used to study the various effects of pile and soil properties on the system.

This study has provided an understanding of system behaviour, and it has also enabled the compilation of tables which give approximately the ground-level displacement and slope for any loaded pile. The tables may be used in the design process as long as the limitations of the soil model are understood and as long as the soil properties can be determined with reasonable accuracy.

A pile characteristic length L_c has been defined and has been shown to be, in general, the most economical length of a laterally loaded pile in a soil of constant properties.

10.1.3 Varying soil properties

Two analyses which allow an elastic isotropic soil to have properties which vary with position have been presented.

The first allows a general variation with position but uses a numerical integration procedure which introduces a number of disadvantages.

The second is an approximate analysis which allows properties to vary with depth, in layers. A computer program has been written to carry out this analysis and it may, like the tables already mentioned, be used in the design process as long as the same conditions are met.

Typical solutions for a two-layer system have been given and it has been noted that effects at ground level are largely dependent on soil properties to a depth of about $\frac{1}{2}L_c$.

10.2 EXPERIMENTAL WORK

10.2.1 Photoelastic investigation

A photoelastic investigation was carried out with the aim of checking the stresses in a two-layer system. Time did not allow the method to be refined sufficiently for accurate results, but the experimental technique is of some interest.

10.2.2 Full-scale test

A full-scale test has been reported. The test procedure proved most satisfactory and two interesting facts emerged from the results:

(1) The soil was linearly elastic up to quite high displacements, and

(2) Soil properties obtained approximately from tests not intended for this purpose correlated reasonably well with actual properties. The indication was that it should be possible to obtain soil elastic properties with reasonable accuracy from simple tests.

10.3 SUGGESTIONS FOR FUTURE RESEARCH

The broad aim of this investigation has been to provide a sound base for future research into some of the various aspects of laterally loaded pile design. It is thus appropriate to conclude by briefly discussing some major areas into which future research should be fruitful.

10.3.1 Inelastic behaviour

Spillers and Stoll[6] and Poulos[7] each modified their elastic analyses by causing the soil to become plastic once a certain yield pressure had been reached. However, the pressure required to cause yielding was determined in a very elementary fashion.

This modification, which can be very easily included in the analysis presented here, has not been made because research at present in progress in the University of Canterbury is aimed at accounting for soil plasticity in a much more realistic way. The aim is to follow system behaviour from the initial elastic stage through to failure.

10.3.2 Dynamic Loading

In New Zealand, a most important agent of lateral load is the earthquake. Thus the consideration of dynamic behaviour is an important avenue for research. In this respect it should be noted that most soils may be expected to behave elastically under dynamic loading.

In the general equation of motion of a dynamically loaded system there will be two unknown quantities: the stiffness of the system and its mass. The stiffness can be found, but the mass will include an unknown vibrating mass of soil. This mass could perhaps be determined from an experimental program.

In connection with earthquake loading, it should be remembered that the displacements are actually applied through the foundation. However, the inertia forces on the structure will have to be resisted by the foundation and it is this aspect of a general interaction problem which can be studied as above.

10.3.3 Pile groups

There will be some interaction among the piles in a closely-spaced group, and the consideration of soil as an elastic continuum should prove useful in accounting for this interaction. The displacement at a point on one pile due to forces applied to the soil through other piles can be calculated. Because of the property of superposition, a total stress system for the group can be obtained to which yield criteria can be applied. Poulos[7] has already presented an analysis of pile groups which uses the elastic half-space soil model.

10.3.4 Determination of soil elastic properties.

The determination of soil elastic properties is of course essential if the analysis presented here is to be useful. The use of geophysical techniques would seem to be very promising, especially if a pile is to be designed to carry dynamic loading. It should be possible also to find elastic properties from some simple soils test such as the triaxial test. Research into this problem would be most useful.

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APPENDIX I

THE MINDLIN SOLUTIONS FOR A POINT FORCE IN AN ELASTIC HALF-SPACE

The Mindlin[5] expressions are reproduced below. u,v,w are the displacements in the x,y,z directions. σ are the direct and τ the shear stresses. The half-space is bound by the plane z = 0, and the force P acts at the point (0,0,c).

$$r = (x^2 + y^2)^{\frac{1}{2}}$$

$$R_1 = (x^2 + y^2 + (z - c)^2)^{\frac{1}{2}} \; ,$$

and $$R_2 = (x^2 + y^2 + (z + c)^2)^{\frac{1}{2}} \; .$$

For a force P in the x direction,

$$u = \frac{P}{16\pi G(1-\mu)}\left[\frac{3-4\mu}{R_1} + \frac{1}{R_2} + \frac{x^2}{R_1^3} + \frac{(3-4\mu)x^2}{R_2^3} + \frac{2cz}{R_2^3}\left(1 - \frac{3x^2}{R_2^2}\right)\right.$$
$$\left. + \frac{4(1-\mu)(1-2\mu)}{R_2 + z + c}\left(1 - \frac{x^2}{R_2(R_2 + z + c)}\right)\right], \qquad (I.1)$$

$$v = \frac{Pxy}{16\pi G(1-\mu)}\left[\frac{1}{R_1^3} + \frac{3-4\mu}{R_2^3} - \frac{6cz}{R_2^5} - \frac{4(1-\mu)(1-2\mu)}{R_2(R_2+z+c)^2}\right], \qquad (I.2)$$

$$w = \frac{Px}{16\pi G(1-\mu)}\left[\frac{z-c}{R_1^3} + \frac{(3-4\mu)(z-c)}{R_2^3} - \frac{6cz(z+c)}{R_2^5}\right.$$
$$\left. + \frac{4(1-\mu)(1-2\mu)}{R_2(R_2+z+c)}\right], \qquad (I.3)$$

$$\sigma_x = \frac{Px}{8\pi(1-\mu)}\left[-\frac{(1-2\mu)}{R_1^3} + \frac{(1-2\mu)(5-4\mu)}{R_2^3} - \frac{3x^2}{R_1^5} - \frac{3(3-4\mu)x^2}{R_2^5}\right.$$

$$- \frac{4(1-\mu)(1-2\mu)}{R_2(R_2+z+c)^2}\left(3 - \frac{x^2(3R_2+z+c)}{R_2^2(R_2+z+c)}\right)$$

$$\left. + \frac{6c}{R_2^5}\left(3c - (3-2\mu)(z+c) + \frac{5x^2 z}{R_2^2}\right)\right], \qquad (I.4)$$

$$\sigma_y = \frac{Px}{8\pi(1-\mu)}\left[\frac{(1-2\mu)}{R_1^3} + \frac{(1-2\mu)(3-4\mu)}{R_2^3} - \frac{3y^2}{R_1^5} - \frac{3(3-4\mu)y^2}{R_2^5}\right.$$

$$- \frac{4(1-\mu)(1-2\mu)}{R_2(R_2+z+c)^2}\left(1 - \frac{y^2(3R_2+z+c)}{R_2^2(R_2+z+c)}\right)$$

$$\left. + \frac{6c}{R_2^5}\left(c - (1-2\mu)(z+c) + \frac{5y^2 z}{R_2^2}\right)\right], \qquad (I.5)$$

$$\sigma_z = \frac{Px}{8\pi(1-\mu)}\left[\frac{(1-2\mu)}{R_1^3} - \frac{(1-2\mu)}{R_2^3} - \frac{3(z-c)^2}{R_1^5} - \frac{3(3-4\mu)(z+c)^2}{R_2^5}\right.$$

$$\left. + \frac{6c}{R_2^5}\left(c + (1-2\mu)(z+c) + \frac{5z(z+c)^2}{R_2^2}\right)\right], \qquad (I.6)$$

$$\tau_{yz} = \frac{Pxy}{8\pi(1-\mu)}\left[-\frac{3(z-c)}{R_1^5} - \frac{3(3-4\mu)(z+c)}{R_2^5}\right.$$

$$\left. + \frac{6c}{R_2^5}\left(1-2\mu + \frac{5z(z+c)}{R_2^2}\right)\right], \qquad (I.7)$$

$$\tau_{zx} = \frac{P}{8\pi(1-\mu)}\left[-\frac{(1-2\mu)(z-c)}{R_1^3} + \frac{(1-2\mu)(z-c)}{R_2^3} - \frac{3x^2(z-c)}{R_1^5}\right.$$

$$\left. - \frac{3(3-4\mu)x^2(z+c)}{R_2^5} - \frac{6c}{R_2^5}\left(z(z+c)-(1-2\mu)x^2 - \frac{5x^2z(z+c)}{R_2^2}\right)\right], \tag{I.8}$$

$$\tau_{xy} = \frac{Py}{8\pi(1-\mu)}\left[-\frac{(1-2\mu)}{R_1^3} + \frac{(1-2\mu)}{R_2^3} - \frac{3x^2}{R_1^5} - \frac{3(3-4\mu)x^2}{R_2^5}\right.$$

$$\left. - \frac{4(1-\mu)(1-2\mu)}{R_2(R_2+z+c)^2}\left(1 - \frac{x^2(3R_2+z+c)}{R_2^2(R_2+z+c)}\right) - \frac{6cz}{R_2^5}\left(1 - \frac{5x^2}{R_2^2}\right)\right]. \tag{I.9}$$

For a force P in the z direction,

$$u,v = \frac{Pr}{16\pi G(1-\mu)}\left[\frac{z-c}{R_1^3} + \frac{(3-4\mu)(z-c)}{R_2^3} - \frac{4(1-\mu)(1-2\mu)}{R_2(R_2+z+c)}\right.$$

$$\left. + \frac{6cz(z+c)}{R_2^5}\right], \tag{I.10}$$

$$w = \frac{P}{16\pi G(1-\mu)}\left[\frac{3-4\mu}{R_1} + \frac{8(1-\mu)^2-(3-4\mu)}{R_2} + \frac{(z-c)^2}{R_1^3}\right.$$

$$\left. + \frac{(3-4\mu)(z+c)^2 - 2cz}{R_2^3} + \frac{6cz(z+c)^2}{R_2^5}\right], \tag{I.11}$$

$$\sigma_x = \frac{P}{8\pi(1-\mu)}\Bigg[\frac{(1-2\mu)(z-c)}{R_1^3} - \frac{3x^2(z-c)}{R_1^5} + \frac{(1-2\mu)\{3(z-c)-4\mu(z+c)\}}{R_2^3}$$

$$- \frac{3(3-4\mu)x^2(z-c)-6c(z+c)\{(1-2\mu)z-2\mu c\}}{R_2^5} - \frac{30cx^2z(z+c)}{R_2^7}$$

$$- \frac{4(1-\mu)(1-2\mu)}{R_2(R_2+z+c)}\left(1 - \frac{x^2}{R_2(R_2+z+c)} - \frac{x^2}{R_2^2}\right)\Bigg], \qquad (I.12)$$

$$\sigma_z = \frac{P}{8\pi(1-\mu)}\Bigg[-\frac{(1-2\mu)(z-c)}{R_1^3} + \frac{(1-2\mu)(z-c)}{R_2^3} - \frac{3(z-c)^3}{R_1^5}$$

$$- \frac{3(3-4\mu)z(z+c)^2 - 3c(z+c)(5z-c)}{R_2^5} - \frac{30cz(z+c)^3}{R_2^7}\Bigg], \qquad (I.13)$$

$$\tau_{yz} = \frac{Py}{8\pi(1-\mu)}\Bigg[-\frac{(1-2\mu)}{R_1^3} + \frac{(1-2\mu)}{R_2^3} - \frac{3(z-c)^2}{R_1^5}$$

$$- \frac{3(3-4\mu)z(z+c)-3c(3z+c)}{R_2^5} - \frac{30cz(z+c)^2}{R_2^7}\Bigg], \qquad (I.14)$$

$$\tau_{xy} = \frac{Pxy}{8\pi(1-\mu)}\Bigg[-\frac{3(z-c)}{R_1^5} - \frac{3(3-4\mu)(z-c)}{R_2^5}$$

$$+ \frac{4(1-\mu)(1-2\mu)}{R_2^2(R_2+z+c)}\left(\frac{1}{R_2+z+c} + \frac{1}{R_2}\right) - \frac{30cz(z+c)}{R_2^7}\Bigg]. \qquad (I.15)$$

Displacements and stresses due to a force in the y direction can be obtained by substituting y for x in equations (I.1) - (I.9) and σ_y and τ_{zx} by making the same substitution in equations (I.12) and (I.14).

APPENDIX II

COMPUTING FACILITIES

During the initial part of this research, the only machine available at the University of Canterbury was an I.B.M. 1620 model 1, with a total available storage of about 29000 decimal digits, or 2900 words. The installation, in October 1967, of an I.B.M. 360 model 44 provided an available core storage capacity of about 8700 single precision (4byte) words, back up storage facilities, and greatly improved speeds of compilation, processing, and output. The still limited core storage capacity made necessary an overlay-type program structure which is described in Appendix III. Although this capacity has been doubled (the available core almost trebled) in September 1969, the overlay structure has been retained, so that the array sizes may be increased.

The programming languages used were Fortran II for the 1620 and Fortran IV for the 360/44. All program listings given are in the latter language and have not yet been modified to use the extra core storage.

APPENDIX III

COMPUTER PROGRAM TO SOLVE FOR A LATERALLY LOADED PILE IN A LAYERED ELASTIC CONTINUUM

The program uses the theory developed in Chapters 3, 6 and 7. It consists of six parts or phases, only one of which is contained in core storage at any time. This structure, shown in fig. III.1, is necessary because of the limited core storage available. Entry to the program is made by one of the statements EXEC FREE, EXEC FIXED, or EXEC STRESS.

EXEC FREE causes the dummy phase FREE, consisting of the single module START, to be placed in core. It sets up constants and then calls in phase BASIC, which contains the two modules BIG and PLATE. BIG reads pile and soil data from cards and solves for a free head pile, the sub-program PLATE calculating on-diagonal terms in the matrix $\mathbf{S}$ (Chapter 3). In the case of a multi-layer soil, BIG uses the basic assumption described in Chapter 6 and transfers control to phase IMPROVE. The layer boundary compatibility correction is then made and control returned to BASIC, for another cycle to be carried out. Printout from BASIC is made when the end-of-cycling condition is met, and control remains with this phase if another free-head pile is to be solved.

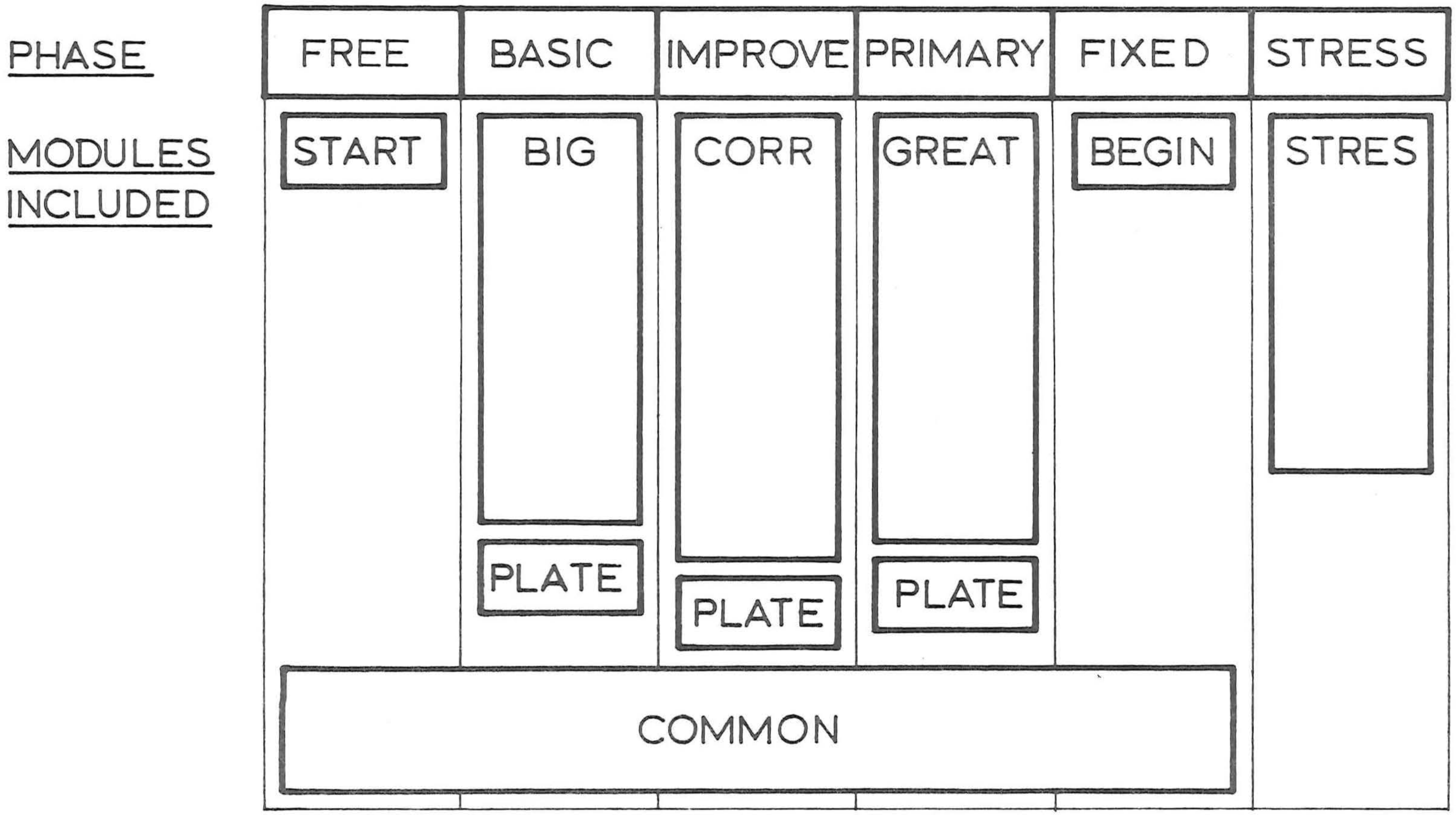

FIG. III.1 - PROGRAM STRUCTURE

EXEC FIXED causes a similar set of instructions to be carried out except that the solution is for a fixed-head pile.

EXEC STRESS causes the module STRES to be placed in core. This should not be done unless a pile solution has just been carried out, for STRES reads some input data which has been placed on disc by the previous phase. Information about the output required is read from cards and output consists of one of the three forms shown in fig. 7.1.

It is not necessary for most jobs that all six phases be loaded. For example, if a free head pile in a soil of constant properties is to be solved, only phases FREE and BASIC are required.

Listings of all modules follows.

```
      MODULE START
C
C     FORMS DUMMY PHASE 'FREE' WHICH SETS UP CONSTANTS TO ALLOW CORRECT
C     TRANSFER BETWEEN PHASES 'BASIC' AND 'IMPROVE' AND THEN CALLS IN
C     'BASIC'
C
      DIMENSION A(67,67),E(67),P(67),U(67),BOUND(10),G(10),PR(10),S11(10
     1),S22(10),S33(10),DU(30),UI(66),B(67,2),C(2,67),R1(2,67),R2(2,2)
      COMMON A,I,Z,NY,NZ,DEL,W,S1,S2,S3,M,DY,E,P,U,BOUND,G,PR,S11,S22,S3
     13,DU,UI,KOUNT,M1,M2,EL,ZMI,ZL,V,ZM,N,NL,X,Y,YX,YZ,ZSH,ZZM,ZN,EIL,K
     2C,J,K,L,KI,KJ,LJ,ZKI,ZKJ,SUM,R3,R4,T,N1,N2,R,ZI,ZJ,MN,DLI,ZMCM,NP,
     3NORUN,KHC,HITE,HU,HL
      NP=0
      KOUNT=1
      KHC=0
      CALL LINK('BASIC')
      END
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      MODULE BIG
C
C     MAIN MODULE FOR PHASE 'BASIC' WHICH SOLVES FOR FREE HEAD PILE
C     USING BASIC ASSUMPTION AND CALLS PHASE 'IMPROVE' IF SOIL HAS MORE
C     THAN ONE LAYER
C
      DIMENSION A(67,67),E(67),P(67),U(67),BOUND(10),G(10),PR(10),S11(10
     1),S22(10),S33(10),DU(30),UI(66),B(67,2),C(2,67),R1(2,67),R2(2,2)
      COMMON A,I,Z,NY,NZ,DEL,W,S1,S2,S3,M,DY,E,P,U,BOUND,G,PR,S11,S22,S3
     13,DU,UI,KOUNT,M1,M2,EL,ZMI,ZL,V,ZM,N,NL,X,Y,YX,YZ,ZSH,ZZM,ZN,EIL,K
     2C,J,K,L,KI,KJ,LJ,ZKI,ZKJ,SUM,R3,R4,T,N1,N2,R,ZI,ZJ,MN,DLI,ZMCM,NP,
     3NORUN,KHC,HITE,HU,HL
      GO TO (500,501),KOUNT
  500 KOUNT=2
      READ(5,876) NORUN
  876 FORMAT(I5)
  503 READ(5,100) EL,ZMI,W,ZL,V,ZM,HITE,MN
  100 FORMAT(7E10.4,I5)
      READ(5,101) NL,(G(I),PR(I),BOUND(I),I=1,NL)
  101 FORMAT(I5/(E10.4,F5.3,F7.1))
      NP=NP+1
      IF(MN.GT.66) MN=66
      KOC=0
      ZM=ZM+V*HITE
      ZMCM=ZM
      ZSH=V
      M=0
  760 M=M+1
      N=MN/M
      ZN=N
      ZZM=M
      IF(ZL/ZN.LT.W/(ZZM+1.)) GO TO 760
      MN=M*N
      DEL=ZL/ZN
      DY=W/ZZM
      NY=10/M*M
      ZNY=NY
      NZ=DEL*ZNY/W+.5
      EIL=6.*EL*ZMI*ZL
      IF(NL.EQ.1) GO TO 301
      DO300I=2,NL
      DO 300J=1,N
      ZJ=J
      Z=(ZJ-.5)*DEL
      IF(BOUND(I).GT.Z) GO TO 300
      IF(BOUND(I).LE.(Z-DEL)) GO TO 300
      BOUND(I)=Z-DEL/2.
  300 CONTINUE
  301 DO21I=1,NL
      S11(I)=3.-4.*PR(I)
      S22(I)=4.-12.*PR(I)+8.*PR(I)**2
   21 S33(I)=50.2656*G(I)*(1.-PR(I))
      KC=1
      M1=M+1
      M2=M*NL
      HU=0.
      HL=0.
      DO201I=1,MN
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```
201 UI(I)=0.
200 KODE=1
 61 DO7I=1,N
    ZI=I
    Z=(ZI-.5)*DEL
    IF(KODE.EQ.2) GO TO 22
    IF(NL-1)30,31,30
 31 S1=S11(1)
    S2=S22(1)
    S3=S33(1)
    GO TO 23
 30 DO20L=2,NL
    IF(Z.GE.BOUND(L)) GO TO 70
    S1=S11(L-1)
    S2=S22(L-1)
    S3=S33(L-1)
    GO TO 23
 70 IF(L.NE.NL) GO TO 20
    S1=S11(NL)
    S2=S22(NL)
    S3=S33(NL)
 20 CONTINUE
 23 CALL PLATE
 22 DO7KI=1,M
    ZKI=KI
    YZ=(ZKI-.5)*DY-W/2.
    LI=(I-1)*M+KI
    DO8J=1,N
    ZJ=J
    X=(ZJ-.5)*DEL
    DO8KJ=1,M
    ZKJ=KJ
    YX=(ZKJ-.5)*DY-W/2.
    LJ=(J-1)*M+KJ
    IF(KODE.EQ.2) GO TO 6
    IF(I.EQ.J) GO TO 6
    Y=YX-YZ
    R3=SQRT(Y**2+(Z-X)**2)
    R4=SQRT(Y**2+(Z+X)**2)
    A(LI,LJ)=(S1/R3+S2/(R4+Z+X)+1./R4+2.*Z*X/R4**3)/S3
  6 R=Z*(ZL-X)*(Z**2+X**2-2.*ZL*X)
    IF(Z.LE.X) A(LI,LJ)=R/EIL-A(LI,LJ)
    IF(Z.GT.X) A(LI,LJ)=(R-ZL*(Z-X)**3)/EIL-A(LI,LJ)
  8 CONTINUE
    IF(KODE-1)7,9,7
  9 B(LI,1)=(ZL-Z)/ZL
    B(LI,2)=Z/ZL
    C(1,LI)=ZL-Z
    C(2,LI)=Z
  7 CONTINUE
    IF(KODE-1)14,15,14
 15 DO13I=1,MN
    T=A(I,I)
    A(I,I)=1.
    DO10J=1,MN
 10 A(I,J)=A(I,J)/T
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```
    DO13K=1,MN
    IF(K-I)11,13,11
 11 T=A(K,I)
    A(K,I)=0.
    DO12J=1,MN
 12 A(K,J)=A(K,J)-T*A(I,J)
 13 CONTINUE
    DO18J=1,MN
    DO18I=1,2
    SUM=0.
    DO19K=1,MN
 19 SUM=SUM+C(I,K)*A(K,J)
 18 R1(I,J)=SUM
    DO16J=1,2
    DO16I=1,2
    SUM=0.
    DO17K=1,MN
 17 SUM=SUM+R1(I,K)*B(K,J)
 16 R2(I,J)=SUM
    RD=R2(1,1)*R2(2,2)-R2(1,2)*R2(2,1)
    R2(1,2)=-R2(1,2)/RD
    R2(2,1)=-R2(2,1)/RD
    R22=R2(2,2)
    R2(2,2)=R2(1,1)/RD
    R2(1,1)=R22/RD
    DO25I=1,N
    ZI=I
    Z=(ZI-.5)*DEL
    DO25K=1,M
    L=(I-1)*M+K
 25 E(L)=(Z*(2*ZL-Z)*(ZL-Z)*ZM)/EIL+UI(L)
    F1=V*ZL+ZM
    F2=-ZM
    SUM=0.
    DO26I=1,MN
 26 SUM=SUM+R1(1,I)*E(I)
    R31=SUM-F1
    SUM=0.
    DO27I=1,MN
 27 SUM=SUM+R1(2,I)*E(I)
    R32=SUM-F2
    DHU=HU
    DHL=HL
    HU=R2(1,1)*R31+R2(1,2)*R32
    HL=R2(2,1)*R31+R2(2,2)*R32
    DHU=ABS((HU-DHU)/HU)
    DHL=ABS((HL-DHL)/HL)
    DO29I=1,MN
    SUM=0.
    DO40K=1,MN
 40 SUM=SUM+A(I,K)*(E(K)-B(K,1)*HU-B(K,2)*HL)
 29 P(I)=SUM
    DO57I=1,MN
    DO57J=1,MN
 57 A(I,J)=0.
    KODE=2
```

```
      GO TO 61
   14 DO62I=1,N
      ZI=I
      Z=(ZI-.5)*DEL
      DO62J=1,M
      L=(I-1)*M+J
      SUM=0.
      DO63K=1,MN
   63 SUM=SUM+A(L,K)*P(K)
   62 U(L)=(ZL-Z)/ZL*HU+Z/ZL*HL+SUM-ZM*Z/EIL*(2.*ZL-Z)*(ZL-Z)
      SLO=0.
      DO 750 I=1,N
      ZI=I
      Z=(ZI-.5)*DEL
      DO 750 J=1,M
      LI=(I-1)*M+J
  750 SLO=SLO+P(LI)*(ZL-Z)*Z/ZL*(2.*(ZL-Z)**2+Z*(3.*ZL-2.*Z))
      SLO=(HU-HL)/ZL+(2.*ZM*ZL**2+SLO)/EIL
      IF(HITE.EQ.0.) GO TO 124
      ZM=ZMOM-V*HITE
      DIS=HU+SLO*HITE+HITE**2/(EL*ZMI)*(ZM/2.+V*HITE/3.)
      SLO=SLO+HITE/(EL*ZMI)*(ZM+V*HITE/2.)
  124 IF(M.EQ.1) GO TO 900
      WRITE(6,903)
  903 FORMAT('1 FORCE DISTRIBUTION'/)
      DO901I=1,N
      L1=(I-1)*M+1
      L2=I*M
  901 WRITE(6,902) (P(L),L=L1,L2)
  902 FORMAT('0'8E12.4)
  900 WRITE(6,103) KC,N,M
  103 FORMAT('1'20X,'PROGRAM VARIABLES       NO OF CYCLES ='I2//44X,'N ='
     1I3//44X,'M ='I2//)
      WRITE(6,104) ZL,W,EL,ZMI
  104 FORMAT(21X,'PILE PROPERTIES',7X,' HEAD FREE'//44X,'LENGTH ='F6.1//
     144X,'WIDTH ='F5.1//44X,'E ='E10.4//44X,'I ='E10.4//)
      WRITE(6,105) HITE,V,ZM
  105 FORMAT(21X,'LOADING DETAILS         HT ABOVE GROUND ='F6.1//44X,'SH
     1EAR ='E10.4//44X,'MOMENT ='E10.4//)
      HITE=-HITE
      WRITE(6,106) NL
  106 FORMAT(21X,'SOIL PROPERTIES',8X,I1,' LAYER SOIL'/)
      IF(NL.EQ.1) WRITE(6,400) G(1),PR(1)
  400 FORMAT(44X,'G ='F7.1//44X,'PR = 'F3.2)
      IF(NL.EQ.1) GO TO 401
      WRITE(6,402)
  402 FORMAT(44X,'DEPTH',6X,'G',7X,'PR'/)
      DO64I=1,NL
      WRITE(6,107) BOUND(I)
  107 FORMAT(43X,F6.1)
   64 WRITE(6,108) G(I),PR(I)
  108 FORMAT(50X,F8.1,4X,F3.2)
  401 WRITE(6,109)
  109 FORMAT(////21X,'DEPTH',6X,'REACTION',6X,'DEFLECTION',6X,'SHEAR',9X
     1,'MOMENT',9X,'SLOPE',10X,'KH'/)
      IF(HITE.EQ.0.) GO TO 120
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      WRITE(6,121) HITE,DIS,V,ZM,SLO
  121 FORMAT(20X,F6.1,F28.4,F14.1,F14.0,E16.4/)
      ZM=ZMOM
      WRITE(6,122) HU,V,ZM
  122 FORMAT(21X,'000.0',F28.4,F14.1,F14.0)
      GO TO 123
  120 WRITE(6,110) HU,V,ZM,SLO
  110 FORMAT(21X,'000.0',F28.4,F14.1,F14.0,E16.4)
  123 DO 65 I=1,N
      ZI=I
      X=(ZI-.5)*DEL
      Z=ZI*DEL
      SUMP=0.
      SUMU=0.
      DO75K=1,M
      L=(I-1)*M+K
      SUMP=SUMP+P(L)
   75 SUMU=SUMU+U(L)
      SUMU=SUMU/ZZM
      V=V-SUMP
      ZM=ZM+V*DEL+.5*SUMP*DEL
      SR=SUMP/(SUMU*DEL)
      L1=(I-1)*M+1
      WRITE(6,111) X,SUMP,U(L1),SR
  111 FORMAT(20X,F6.1,F13.1,F15.4,40X,F15.1)
      IF(Z.NE.ZL) WRITE(6,112) Z,V,ZM
  112 FORMAT(20X,F6.1,25X,F17.1,F14.0)
   65 CONTINUE
      WRITE(6,113) ZL,HL,V,ZM
  113 FORMAT(20X,F6.1,F28.4,F14.1,F14.0//////////)
      REWIND 1
      WRITE(1) MN,(P(I),I=1,MN),NL,(G(I),PR(I),BOUND(I),I=1,NL),ZL,W,MN,
     1M,N,M1,M2,(DU(I),I=M1,M2)
      IF(NL.EQ.1) GO TO 502
      IF(KC.EQ.7) GO TO 502
      IF(DHU.GT..01) GO TO 802
      IF(DHL.LE..05) GO TO 502
  802 ZM=ZMOM
      V=ZSH
      HITE=-HITE
      CALL LINK('IMPROVE')
  501 KC=KC+1
      GO TO 200
  502 IF(NP.LT.NORUN) GO TO 503
      CONTINUE
      END
```

```
      MODULE BEGIN

C     FORMS DUMMY PHASE 'FIXED' WHICH IS SIMILAR TO 'FREE' BUT WHICH
C     CALLS IN 'PRIMARY'
C
      DIMENSION A(67,67),E(67),P(67),U(67),BOUND(10),G(10),PR(10),S11(10
     1),S22(10),S33(10),DU(30),UI(66),B(67,2),C(2,67),R1(2,67),R2(2,2)
      COMMON A,I,Z,NY,NZ,DEL,W,S1,S2,S3,M,DY,E,P,U,BOUND,G,PR,S11,S22,S3
     13,DU,UI,KOUNT,M1,M2,EL,ZMI,ZL,V,ZM,N,NL,X,Y,YX,YZ,ZSH,ZZM,ZN,EIL,K
     2C,J,K,L,KI,KJ,LJ,ZKI,ZKJ,SUM,R3,R4,T,N1,N2,R,ZI,ZJ,MN,DUI,ZMCM,NP,
     3NORUN,KHC,HITE,HU,HL
      NP=0
      KOUNT=1
      KHC=1
      CALL LINK('PRIMARY')
      END
```

```
      MODULE GREAT

C     MAIN MODULE FOR PHASE 'PRIMARY'. SIMILAR TO MODULE 'BIG' BUT
C     SOLVES FOR A FIXED HEAD PILE.
C
      DIMENSION A(67,67),E(67),P(67),U(67),BOUND(10),G(10),PR(10),S11(10
     1),S22(10),S33(10),DU(30),UI(66),B(67,2),C(2,67),R1(2,67),R2(2,2)
      COMMON A,I,Z,NY,NZ,DEL,W,S1,S2,S3,M,DY,E,P,U,BOUND,G,PR,S11,S22,S3
     13,DU,UI,KOUNT,M1,M2,EL,ZMI,ZL,V,ZM,N,NL,X,Y,YX,YZ,ZSH,ZZM,ZN,EIL,K
     2C,J,K,L,KI,KJ,LJ,ZKI,ZKJ,SUM,R3,R4,T,N1,N2,R,ZI,ZJ,MN,DUI,ZMCM,NP,
     3NORUN,KHC,HITE,HU,HL
      GO TO (500,501),KOUNT
  500 KOUNT=2
      READ(5,876) NORUN
  876 FORMAT(I5)
  503 READ(5,100) EL,ZMI,W,ZL,V,HITE,MN
  100 FORMAT(6E10.4,I5)
      READ(5,101) NL,(G(I),PR(I),BOUND(I),I=1,NL)
  101 FORMAT(I5/(E10.4,F5.3,F7.1))
      NP=NP+1
      IF(MN.GT.66) MN=66
      KOC=0
      ZSH=V
      M=0
  760 M=M+1
      N=MN/M
      ZN=N
      ZZM=M
      IF(ZL/ZN.LT.W/(ZZM+1.)) GO TO 760
      MN=M*N
      DEL=ZL/ZN
      DY=W/ZZM
      NY=10/M*M
      ZNY=NY
      NZ=DEL*ZNY/W+.5
      EIL=6.*EL*ZMI*ZL
      IF(NL.EQ.1) GO TO 301
      DO 300 I=2,NL
      DO 300 J=1,N
      ZJ=J
      Z=(ZJ-.5)*DEL
      IF(BOUND(I).GT.Z) GO TO 300
      IF(BOUND(I).LE.(Z-DEL)) GO TO 300
      BOUND(I)=Z-DEL/2.
  300 CONTINUE
  301 DO 21 I=1,NL
      S11(I)=3.-4.*PR(I)
      S22(I)=4.-12.*PR(I)+8.*PR(I)**2
   21 S33(I)=50.2656*G(I)*(1.-PR(I))
      KC=1
      M1=M+1
      M2=M*NL
      HU=0.
      HL=0.
      DO 201 I=1,MN
  201 UI(I)=0.
  200 KODE=1
      MN1=MN+1
```

```
61 DO 7 I=1,N
   ZI=I
   Z=(ZI-.5)*DEL
   IF(NL-1) 30,31,30
31 S1=S11(1)
   S2=S22(1)
   S3=S33(1)
   GO TO 23
30 DO 20 L=2,NL
   IF(Z.GE.BOUND(L)) GO TO 70
   S1=S11(L-1)
   S2=S22(L-1)
   S3=S33(L-1)
   GO TO 23
70 IF(L.NE.NL) GO TO 20
   S1=S11(NL)
   S2=S22(NL)
   S3=S33(NL)
20 CONTINUE
23 CALL PLATE
   DO 7 KI=1,M
   ZKI=KI
   YZ=(ZKI-.5)*DY-W/2.
   LI=(I-1)*M+KI
   DO 8 J=1,N
   ZJ=J
   X=(ZJ-.5)*DEL
   DO 8 KJ=1,M
   ZKJ=KJ
   YX=(ZKJ-.5)*DY-W/2.
   LJ=(J-1)*M+KJ
   IF(I.EQ.J) GO TO 22
   Y=YX-YZ
   R3=SQRT(Y**2+(Z-X)**2)
   R4=SQRT(Y**2+(Z+X)**2)
   A(LI,LJ)=(S1/R3+S2/(R4+Z+X)+1./R4+2.*Z*X/R4**3)/S3
22 IF(KCODE-1) 8,6,8
 6 R=Z*(ZL-X)*(Z**2+X**2-2.*ZL*X)
   IF(Z.LE.X) A(LI,LJ)=R/EIL-A(LI,LJ)
   IF(Z.GT.X) A(LI,LJ)=(R-ZL*(Z-X)**3)/EIL-A(LI,LJ)
 8 CONTINUE
   IF(KCODE-1) 7,9,7
 9 B(LI,1)=(ZL-Z)/ZL
   B(LI,2)=Z/ZL
   C(1,LI)=ZL-Z
   C(2,LI)=Z
 7 CONTINUE
   IF(KCODE-1) 14,15,14
15 DO 25 I=1,N
   ZI=I
   Z=(ZI-.5)*DEL
   DO 25 K=1,M
   L=(I-1)*M+K
   A(L,MN1)=-(Z*(2*ZL-Z)*(ZL-Z))/EIL
   E(L)=UI(L)
25 A(MN1,L)=(ZL-Z)*Z/ZL*(2.*(ZL-Z)**2+Z*(3.*ZL-2.*Z))/EIL
```



```
   A(MN1,MN1)=2.*ZL**2/EIL+HITE/(EL*ZMI)
   B(MN1,1)=1./ZL
   B(MN1,2)=-1./ZL
   C(1,MN1)=-1.
   C(2,MN1)=1.
   E(MN1)=V*HITE**2/(2.*EL*ZMI)
   DO 13 I=1,MN1
   T=A(I,I)
   A(I,I)=1.
   DO 10 J=1,MN1
10 A(I,J)=A(I,J)/T
   DO 13 K=1,MN1
   IF(K-I) 11,13,11
11 T=A(K,I)
   A(K,I)=0.
   DO 12 J=1,MN1
12 A(K,J)=A(K,J)-T*A(I,J)
13 CONTINUE
   DO 18 J=1,MN1
   DO 18 I=1,2
   SUM=0.
   DO 19 K=1,MN1
19 SUM=SUM+C(I,K)*A(K,J)
18 R1(I,J)=SUM
   DO 16 J=1,2
   DO 16 I=1,2
   SUM=0.
   DO 17 K=1,MN1
17 SUM=SUM+R1(I,K)*B(K,J)
16 R2(I,J)=SUM
   RD=R2(1,1)*R2(2,2)-R2(1,2)*R2(2,1)
   R2(1,2)=-R2(1,2)/RD
   R2(2,1)=-R2(2,1)/RD
   R22=R2(2,2)
   R2(2,2)=R2(1,1)/RD
   R2(1,1)=R22/RD
   SUM=0.
   DO 26 I=1,MN1
26 SUM=SUM+R1(1,I)*E(I)
   R31=SUM-V*ZL
   SUM=0.
   DO 27 I=1,MN1
27 SUM=SUM+R1(2,I)*E(I)
   R32=SUM
   DHU=HU
   DHL=HL
   HU=R2(1,1)*R31+R2(1,2)*R32
   HL=R2(2,1)*R31+R2(2,2)*R32
   DHU=ABS((HU-DHU)/HU)
   DHL=ABS((HL-DHL)/HL)
   DO 29 I=1,MN1
   SUM=0.
   DO 40 K=1,MN1
40 SUM=SUM+A(I,K)*(E(K)-B(K,1)*HU-B(K,2)*HL)
29 P(I)=SUM
   ZM=P(MN1)
```

```
      ZMOM=ZM
      KODE=2
      GO TO 61
   14 DO 62 I=1,MN
      SUM=0.
      DO 63 J=1,MN
   63 SUM=SUM+A(I,J)*P(J)
   62 U(I)=SUM+UI(I)
      IF(HITE.EQ.0.) GO TO 124
      SLO=0.
      DO 750 I=1,N
      ZI=I
      Z=(ZI-.5)*DEL
      DO 750 J=1,M
      LI=(I-1)*M+J
  750 SLO=SLO+P(LI)*(ZL-Z)*Z/ZL*(2.*(ZL-Z)**2+Z*(3.*ZL-2.*Z))
      SLO=(HU-HL)/ZL+(2.*ZM*ZL**2+SLO)/EIL
      DIS=HU+SLO*HITE+HITE**2/(EL*ZMI)*(ZM/2.-V*HITE/6.)
      ZM=ZMOM-V*HITE
  124 IF(M.EQ.1) GO TO 900
      WRITE(6,903)
  903 FORMAT('1 FORCE DISTRIBUTION'/)
      DO 901 I=1,N
      L1=(I-1)*M+1
      L2=I*M
  901 WRITE(6,902) (P(L),L=L1,L2)
  902 FORMAT('0'8E12.4)
  900 WRITE(6,103) KC,N,M
  103 FORMAT('1'20X,'PROGRAM VARIABLES       NO OF CYCLES ='I2//44X,'N ='
     1I3//44X,'M ='I2//)
      WRITE(6,104) ZL,W,EL,ZMI
  104 FORMAT(21X,'PILE PROPERTIES',8X,'HEAD FIXED'//44X,'LENGTH ='F6.1//
     144X,'WIDTH ='F5.1//44X,'E ='E9.4//44X,'I ='E9.4//)
      WRITE(6,105) HITE,V,ZM
  105 FORMAT(21X,'LOADING DETAILS         HT ABOVE GROUND ='F6.1//44X,'SH
     1EAR ='E10.4//44X,'FIXING MOMENT ='E11.4//)
      HITE=-HITE
      WRITE(6,106) NL
  106 FORMAT(21X,'SOIL PROPERTIES',8X,I1,' LAYER SOIL'/)
      IF(NL.EQ.1) WRITE(6,400) G(1),PR(1)
  400 FORMAT(44X,'G ='F7.1//44X,'PR =  'F3.2)
      IF(NL.EQ.1) GO TO 401
      WRITE(6,402)
  402 FORMAT(44X,'DEPTH',6X,'G',7X,'PR'/)
      DO 64 I=1,NL
      WRITE(6,107) BOUND(I)
  107 FORMAT(43X,F6.1)
   64 WRITE(6,108) G(I),PR(I)
  108 FORMAT(50X,F8.1,4X,F3.2)
  401 WRITE(6,109)
  109 FORMAT(////21X,'DEPTH',6X,'REACTION',6X,'DEFLECTION',6X,'SHEAR',9X
     1,'MOMENT',8X,'KH'/)
      IF(HITE.EQ.0.) GO TO 120
      WRITE(6,121) HITE,DIS,V,ZM
  121 FORMAT(20X,F6.1,F28.4,F14.1,F14.0/)
      ZM=ZMOM
```



```
  120 WRITE(6,122) HU,V,ZM
  122 FORMAT(21X,'000.0',F28.4,F14.1,F14.0)
      DO 65 I=1,N
      ZI=I
      X=(ZI-.5)*DEL
      Z=ZI*DEL
      SUMP=0.
      SUMU=0.
      DO 75 K=1,M
      L=(I-1)*M+K
      SUMP=SUMP+P(L)
   75 SUMU=SUMU+U(L)
      SUMU=SUMU/ZZM
      V=V-SUMP
      ZM=ZM+V*DEL+.5*SUMP*DEL
      SR=SUMP/(SUMU*DEL)
      L1=(I-1)*M+1
      WRITE(6,111) X,SUMP,U(L1),SR
  111 FORMAT(20X,F6.1,F13.1,F15.4,F40.1)
      IF(Z.NE.ZL) WRITE(6,112) Z,V,ZM
  112 FORMAT(20X,F6.1,25X,F17.1,F14.0)
   65 CONTINUE
      WRITE(6,113) ZL,HL,V,ZM
  113 FORMAT(20X,F6.1,F28.4,F14.1,F14.0/////////)
      REWIND 1
      WRITE(1)MN,(P(I),I=1,MN),NL,(G(I),PR(I),BOUND(I),I=1,NL),ZL,W,MN,M
     1,N,M1,M2,(DU(I),I=M1,M2)
      IF(NL.EQ.1) GO TO 502
      IF(KC.EQ.7) GO TO 502
      IF(DHU.GT..01) GO TO 802
      IF(DHL.LE..05) GO TO 502
  802 ZM=ZMOM
      V=ZSH
      HITE=-HITE
      CALL LINK('IMPROVE')
  501 KC=KC+1
      GO TO 200
  502 IF(NP.LT.NORUN) GO TO 503
      CONTINUE
      END
```

```
C     MAIN MODULE FOR PHASE 'IMPROVE' WHICH APPLIES LAYER BOUNDARY
C     COMPATIBILITY CORRECTION, ASSEMBLES MATRIX UC, AND RETURNS CONTROL
C     TO CALLING PHASE.
C
      DIMENSION A(67,67),E(67),P(67),U(67),BOUND(10),G(10),PR(10),S11(10
     1),S22(10),S33(10),DU(30),UI(66),A1(30,30),A2(30,30)
      COMMON A,I,Z,NY,NZ,DEL,W,S1,S2,S3,M,DY,F,P,U,BOUND,G,PR,S11,S22,S3
     13,DU,UI,KOUNT,M1,M2,EL,ZMI,ZL,V,ZM,N,NL,X,Y,YX,YZ,ZSH,ZZM,ZN,EIL,K
     2C,J,K,L,KI,KJ,LJ,ZKI,ZKJ,SUM,R3,R4,T,N1,N2,R,ZI,ZJ,MN,DUI,ZMCM,NP,
     3NORUN,KHC,HITE,HU,HL
C
C     CALCULATE LACK OF COMPATIBILITY AT LAYER BOUNDARIES
C
      DO1I=2,NL
      Z=BOUND(I)
      DO1KI=1,M
      LI=(I-1)*M+KI
      ZKI=KI
      YZ=(ZKI-.5)*DY-W/2.
      SUMA=0.
      SUMB=0.
      DO2J=1,N
      ZJ=J
      X=(ZJ-.5)*DEL
      DO2KJ=1,M
      ZKJ=KJ
      YX=(ZKJ-.5)*DY-W/2.
      Y=YX-YZ
      LJ=(J-1)*M+KJ
      LK=(I-1)*M+KJ
      R3=SQRT(Y**2+(Z-X)**2)
      R4=SQRT(Y**2+(Z+X)**2)
      K=2
    3 IK=I+1-K
      S1=S11(IK)
      S2=S22(IK)
      S3=S33(IK)
      IF(ABS(Z-X).GE.DEL/3.) GO TO 71
      IF(KJ.EQ.1) CALL PLATE
      ST=A(LI,LK)
      GO TO 72
   71 ST=(S1/R3+S2/(R4+Z+X)+1./R4+2.*Z*X/R4**3)/S3
   72 IF(K.EQ.1) GO TO 2
      SUMA=SUMA+P(LJ)*ST
      K=1
      GO TO 3
    2 SUMB=SUMB+P(LJ)*ST
    1 DU(LI)=SUMA-SUMB
C
C     CALCULATE FORCES TO ELIMINATE LACK OF COMPATIBILITY
C
      DO9I=M1,M2
      DO9J=M1,M2
      A1(I,J)=0.
      A2(I,J)=0.
    9 A(I,J)=0.
```



```
      DO10I=2,NL
      Z=BOUND(I)
      DO10KI=1,M
      ZKI=KI
      LI=(I-1)*M+KI
      YZ=(ZKI-.5)*DY-W/2.
      K=2
   13 J1=I+1-K
      S1=S11(J1)
      S2=S22(J1)
      S3=S33(J1)
      J2=J1+1
      IF(I.EQ.2) J1=2
      IF(I.EQ.NL) J2=NL
      DO11J=J1,J2
      X=BOUND(J)
      DO11KJ=1,M
      ZKJ=KJ
      YX=(ZKJ-.5)*DY-W/2.
      LJ=(J-1)*M+KJ
      LK=(I-1)*M+KJ
      Y=YX-YZ
      R3=SQRT(Y**2+(Z-X)**2)
      R4=SQRT(Y**2+(Z+X)**2)
      IF(ABS(Z-X).GE.DEL/3.) GO TO 12
      IF(KJ.EQ.1) CALL PLATE
      ST=A(LI,LK)
      GO TO 73
   12 ST=(S1/R3+S2/(R4+Z+X)+1./R4+2.*Z*X/R4**3)/S3
   73 IF(K.EQ.2) A1(LI,LJ)=ST
      IF(K.EQ.1) A2(LI,LJ)=ST
   11 CONTINUE
      IF(K.EQ.1) GO TO 10
      K=1
      GO TO 13
   10 CONTINUE
      DO21I=M1,M2
      DO21J=M1,M2
   21 A(I,J)=A1(I,J)+A2(I,J)
      DO33I=M1,M2
      T=A(I,I)
      A(I,I)=1.
      DO30J=M1,M2
   30 A(I,J)=A(I,J)/T
      DO33K=M1,M2
      IF(K-I)31,33,31
   31 T=A(K,I)
      A(K,I)=0.
      DO32J=M1,M2
   32 A(K,J)=A(K,J)-T*A(I,J)
   33 CONTINUE
      DO38I=M1,M2
      SUM=0.
      DO39K=M1,M2
   39 SUM=SUM+A(I,K)*DU(K)
   38 P(I)=SUM
```

```
C
C     CALCULATE THE DEFLECTIONS DUE TO CORRECTING FORCES
C
      DO41I=1,NL
      N1=BOUND(I)/DEL+1
      IF(N1.GT.N) GO TO 41
      IF(I.NE.NL) N2=BOUND(I+1)/DEL
      IF(I.EQ.NL) N2=N
      IF(N2.GT.N) N2=N
      S1=S11(I)
      S2=S22(I)
      S3=S33(I)
      DO41J=N1,N2
      ZJ=J
      Z=(ZJ-.5)*DEL
      DO41KJ=1,M
      ZKJ=KJ
      YZ=(ZKJ-.5)*DY-W/2.
      LI=(J-1)*M+KJ
      LJ=(I-1)*M+KJ
      J1=I
      J2=I+1
      IF(I.EQ.1) J1=I+1
      IF(I.EQ.NL) J2=I
      SUM=0.
      DO42K=J1,J2
      X=BOUND(K)
      DO42KK=1,M
      ZKK=KK
      YX=(ZKK-.5)*DY-W/2.
      Y=YX-YZ
      LK=(K-1)*M+KK
      LL=(I-1)*M+KK
      R3=SQRT(Y**2+(Z-X)**2)
      R4=SQRT(Y**2+(Z+X)**2)
      IF(ABS(Z-X).GE.DEL/3.) GO TO 43
      IF(KK.EQ.1) CALL PLATE
      ST=A(LJ,LL)
      GO TO 44
   43 ST=(S1/R3+S2/(R4+Z+X)+1./R4+2.*Z*X/R4**3)/S3
   44 IF(Z.GT.X) SUM=SUM+P(LK)*ST
      IF(Z.LE.X) SUM=SUM-P(LK)*ST
   42 CONTINUE
      UI(LI)=SUM
   41 CONTINUE
      DO 50I=M1,M2
   50 DU(I)=P(I)
      IF(KHC.EQ.0) CALL LINK('BASIC')
      IF(KHC.EQ.1) CALL LINK('PRIMARY')
      END
```

```
C     SUBPROGRAM FOR MODULES 'BIG', 'GREAT', AND 'CORR'. CALCULATES
C     ON-DIAGONAL TERMS IN FLEXIBILITY MATRIX S
C
      SUBROUTINE PLATE
C
C     ASSUMES A FORCE DISTRIBUTION OVER THE RECTANGULAR AREA DZ * WIDTH
C     OF PILE WHICH MAY VARY WITH Y BUT IS CONSTANT WITH Z.
C
      DIMENSION B(67,67),A(10,10),P(10)
      COMMON B,I,Z,NY,NZ,DEL,W,S1,S2,S3,M,DY
      L=I
      ZNY=NY
      ZNZ=NZ
      DDY=W/ZNY
      DDZ=DEL/ZNZ
C
C     TO CALCULATE THE DISTRIBUTION OF FORCE ALONG A LINE PARALLEL TO
C     THE Y-AXIS WHICH HAS A CONSTANT DEFLECTION
C
      DO1I=1,NY
      ZI=I
      Y1=(ZI-.5)*DDY-W/2.
      DO1J=1,NY
      ZJ=J
      Y2=(ZJ-.5)*DDY-W/2.
      R1=ABS(Y1-Y2)
      R2=SQRT(R1**2+4.*Z**2)
      IF(I.EQ.J) R1=.285*SQRT(DDY*DDZ)
    1 A(I,J)=(S1/R1+S2/(R2+2.*Z)+1./R2+2.*Z*Z/R2**3)/S3
      DO3I=1,NY
      T=A(I,I)
      A(I,I)=1.
      DO5J=1,NY
    5 A(I,J)=A(I,J)/T
      DO3K=1,NY
      IF(K-I)4,3,4
    4 T=A(K,I)
      A(K,I)=0.
      DO2J=1,NY
    2 A(K,J)=A(K,J)-T*A(I,J)
    3 CONTINUE
      DO6I=1,NY
      SUM=0.
      DO7J=1,NY
    7 SUM=SUM+A(I,J)
    6 P(I)=SUM
C
C     TO CALCULATE FLEXIBILITY COEFFICIENT FOR AREA DZ*DY
C
      I=L
      DO20KI=1,M
      ZKI=KI
      LI=(I-1)*M+KI
      DO20KJ=1,M
      ZKJ=KJ
      LJ=(I-1)*M+KJ
```



```
      U=0.
      N1=(KI-1)*NY/M+1
      N2=KI*NY/M
      SUM=0.
      DO10L=1,NZ
      ZL=L
      C=(ZL-.5)*DDZ+Z-DEL/2.
      DO 10J=N1,N2
      ZJ=J
      SUM=SUM+P(J)
      Y=(ZKJ-2.*ZKI+.5)*DY+(ZJ-.5)*DDY
      R1=SQRT(Y**2+(Z-C)**2)
      R2=SQRT(Y**2+(Z+C)**2)
      IF(R1.LT..001*W) R1=.285*SQRT(DDY*DDZ)
   10 U=U+P(J)/S3*(S1/R1+S2/(R2+Z+C)+1./R2+2.*Z*C/R2**3)
   20 B(LI,LJ)=U/SUM
      RETURN
      END
```

```
C      FORMS PHASE 'STRESS' WHICH COMPUTES SOIL STRESSES
C
      DIMENSION P(66),Q(10),G(10),RP(10),BOUND(10),EL(3),EM(3),EN(3)
      DSX(X,Y,Z,C)=(-A1/R1**3&A1*(5.-4.*PR)/R2**3-3.*X**2/R1**5-3.*A2*X*
     1*2/R2**5-4.*A3*A1/(R2*(R2&A4)**2)*(3.-X**2*A6)&6.*C/R2**5*(3.*C-(3
     2.-2.*PR)*A4&5.*X**2*Z/R2**2))*X/(25.1328*A3)
      DSY(X,Y,Z,C)=(A1/R1**3&A1*A2/R2**3-3.*Y**2/R1**5-3.*A2*Y**2/R2**5-
     14.*A3*A1/(R2*(R2&A4)**2)*(1.-Y**2*A6)&6.*C/R2**5*(C-A1*A4&5.*Z*Y**
     22/R2**2))*X/(25.1328*A3)
      DSZ(X,Y,Z,C)=(A1/R1**3-A1/R2**3-3.*A5**2/R1**5-3.*A2*A4**2/R2**5&6
     1.*C/R2**5*(C&A1*A4&5.*Z*A4**2/R2**2))*X/(25.1328*A3)
      SSYZ(X,Y,Z,C)=(-3.*A5/R1**5-3.*A2*A4/R2**5&6.*C/R2**5*(A1&5.*Z*A4/
     1R2**2))*X*Y/(25.1328*A3)
      SSZX(X,Y,Z,C)=(-A1*A5/R1**3&A1*A5/R2**3-3.*X**2*A5/R1**5-3.*A2*A4*
     1X**2/R2**5-6.*C/R2**5*(Z*A4-A1*X**2-5.*Z*A4*X**2/R2**2))/(25.1328*
     2A3)
      SSXY(X,Y,Z,C)=(-A1/R1**3&A1/R2**3-3.*X**2/R1**5-3.*A2*X**2/R2**5-4
     1.*A3*A1/     (R2*(R2&A4)**2)*(1.-X**2*A6)-6.*C*Z/R2**5*(1.-5.*X**2/
     2R2**2))*Y/(25.1328*A3)
      READ(5,101) XMIN,XMAX,YMIN,YMAX,ZMIN,ZMAX,MX,MY,MZ,K1,K2,K3
  101 FORMAT(6F8.3,6I4)
      REWIND 1
      READ(1) MN,(P(I),I=1,MN),NL,(G(I),RP(I),BOUND(I),I=1,NL),ZL,W,MN,M
     1,N,M1,M2,(Q(I),I=M1,M2)
      ZN=N
      ZMX=MX
      ZMY=MY
      ZMZ=MZ
      ZZM=M
      IF(K1.EQ.0) GO TO 63
      KODE=1
      WRITE(6,108)
      WRITE(6,109)
  108 FORMAT(14H1      POSITION,20X,15HDIRECT STRESSES,23X,14HSHEAR STRES
     1SES)
  109 FORMAT(5H0    X,5X,1HY,5X,1HZ,10X,2HXX,10X,2HYY,10X,2HZZ,12X,2HYZ,1
     10X,2HZX,10X,2HXY)
      GO TO 62
   63 IF(K2.EQ.0) GO TO 64
      KODE=2
      WRITE(6,120)
      WRITE(6,121)
  120 FORMAT(14H1      POSITION,33X,39HPRINCIPAL STRESSES AND THEIR DIREC
     1TIONS)
  121 FORMAT(5H0    X,5X,1HY,5X,1HZ,9X,2HP1,7X,2HL1,4X,2HM1,4X,2HN1,8X,2H
     1P2,7X,2HL2,4X,2HM2,4X,2HN2,8X,2HP3,7X,2HL3,4X,2HM3,4X,2HN3)
      GO TO 62
   64 IF(K3.EQ.0) GO TO 20
      KODE=3
      WRITE(6,130)
      WRITE(6,131)
  130 FORMAT(14H1      POSITION11X,64HDIFFERENCE BETWEEN PRINCIPAL STRESS
     1ES AND DIRECTION OF MAX. P.S.)
  131 FORMAT(5H0    X,5X,1HY,5X,1HZ,13X,8HXY PLANE,15X,8HYZ PLANE,15X,8HZ
     5X PLANE)
   62 DO 2 I=1,MX
```



```
      ZI=I
      IF(MX.EQ.1) X=XMIN
      IF(MX.NE.1) X=XMIN+(ZI-1.)*(XMAX-XMIN)/(ZMX-1.)
      DO2J=1,MY
      ZJ=J
      IF(MY.EQ.1)Y1=YMIN
      IF(MY.NE.1)Y1=YMIN+(ZJ-1.)*(YMAX-YMIN)/(ZMY-1.)
      DO2K=1,MZ
      ZK=K
      IF(MZ.EQ.1) Z=ZMIN
      IF(MZ.NE.1) Z=ZMIN+(ZK-1.)*(ZMAX-ZMIN)/(ZMZ-1.)
      XX=0.
      YY=0.
      ZZ=0.
      YZ=0.
      ZX=0.
      XY=0.
      DO9JJ=1,N
      ZJJ=JJ
      C=(ZJJ-.5)*ZL/ZN
      DO9KK=1,M
      ZKK=KK
      Y2=(ZKK-.5)*W/ZZM-W/2.
      Y=Y1-Y2
      II=(JJ-1)*M&KK
      IF(ABS(Z-C).GT..0001) GO TO 65
      IF(ABS(X).GT..0001) GO TO 65
      IF(ABS(Y).GT..0001) GO TO 65
      WRITE(6,66) X,Y,Z
   66 FORMAT(1H0,3F6.1,21H     STRESSES INFINITE)
      GO TO 2
   65 A4=Z&C
      A5=Z-C
      R1=SQRT(X**2&Y**2&A5**2)
      R2=SQRT(X**2&Y**2&A4**2)
      A6=(3.*R2&A4)/(R2**2*(R2&A4))
      QQ=P(II)
      IF(NL.EQ.1) L=1
      IF(NL.EQ.1) GO TO 12
      DO 10 II=2,NL
      IF(Z.GE.BOUND(II)) GO TO 11
      L=II-1
      GO TO 12
   11 IF(II.EQ.NL) L=NL
   10 CONTINUE
   12 PR=RP(L)
      A1=1.-2.*PR
      A2=3.-4.*PR
      A3=1.-PR
      XX=XX&QQ*DSX(X,Y,Z,C)
      YY=YY&QQ*DSY(X,Y,Z,C)
      ZZ=ZZ&QQ*DSZ(X,Y,Z,C)
      YZ=YZ&QQ*SSYZ(X,Y,Z,C)
      ZX=ZX&QQ*SSZX(X,Y,Z,C)
    9 XY=XY&QQ*SSXY(X,Y,Z,C)
      KKK=1
```

```
      IF(NL.EQ.1) GO TO 41
      IF(L.EQ.1) GO TO 13
      CD=RCUND(L)
      LL=L
   16 DO 14 JJ=1,M
      KK=(LL-1)*M&JJ
      ZJJ=JJ
      YD=-W/2.&(ZJJ-.5)*W/ZZM
      YD=Y-YD
      A4=Z&CD
      A5=Z-CD
      R1=SQRT(X**2&YD**2&A5**2)
      R2=SQRT(X**2&YD**2&A4**2)
      A6=(3.*R2&A4)/(R2**2*(R2&A4))
      IF(KKK.EQ.1) QQ=-Q(KK)
      IF(KKK.EQ.2) QQ=Q(KK)
      XX=XX&QQ*DSX(X,YD,Z,CD)
      YY=YY&QQ*DSY(X,YD,Z,CD)
      ZZ=ZZ&QQ*DSZ(X,YD,Z,CD)
      YZ=YZ&QQ*SSYZ(X,YD,Z,CD)
      ZX=ZX&QQ*SSZX(X,YD,Z,CD)
   14 XY=XY&QQ*SSXY(X,YD,Z,CD)
      IF(KKK.EQ.2) GO TO 41
   13 IF(L.EQ.NL) GO TO 41
      CD=RCUND(L&1)
      LL=L&1
      KKK=2
      GO TO 16
   41 IF(KCDE.NE.1) GO TO 42
      WRITE(6,110) X,Y1,Z,XX,YY,ZZ,YZ,ZX,XY
  110 FORMAT(1H0,3F6.1,2X,3E12.4,2X,3E12.4)
      GO TO 2
   42 IF(KCDE.EQ.3) GO TO 200
      AA=-XX-YY-ZZ
      BB=YY*ZZ&ZZ*XX&XX*YY-YZ**2-ZX**2-XY**2
      CC=-XX*YY*ZZ-2.*YZ*ZX*XY
      CC=CC&XX*YZ**2&YY*ZX**2&ZZ*XY**2
      PS=ZX
   61 FPS=PS**3&AA*PS**2&BB*PS&CC
      FFPS=3.*PS**2&2.*AA*PS&BB
      DPS=-FPS/FFPS
      PS=PS&DPS
      IF(ABS(DPS)-ABS(PS)/10000.)60,60,61
   60 DD=AA&PS
      EE=BB&DD*PS
      PS2=.5*(-DD&SQRT(DD**2-4.*EE))
      PS3=.5*(-DD-SQRT(DD**2-4.*EE))
      IF(ABS(PS)-ABS(PS2))180,180,181
  181 IF(ABS(PS)-ABS(PS3))182,182,183
  183 PS1=PS
      GO TO 190
  180 IF(ABS(PS2)-ABS(PS3))182,182,185
  185 PS1=PS2
      PS2=PS
      GO TO 190
  182 PS1=PS3
```



```
      PS3=PS
  190 DO603KS=1,3
      IF(KS.EQ.1) PS=PS1
      IF(KS.EQ.2) PS=PS2
      IF(KS.EQ.3) PS=PS3
      IF(PS.NE.YY) GO TO 606
      EL(KS)=0.
      EM(KS)=1.
      EN(KS)=0.
      GO TO 603
  606 IF(PS.NE.ZZ) GO TO 605
      EL(KS)=0.
      EM(KS)=0.
      EN(KS)=1.
      GO TO 603
  605 EL(KS)=(YY-PS)*(ZZ-PS)-YZ*YZ
      EM(KS)=YZ*ZX-XY*(ZZ-PS)
      EN(KS)=XY*YZ-ZX*(YY-PS)
      EK=SQRT(EL(KS)**2+EM(KS)**2+EN(KS)**2)
      EL(KS)=EL(KS)/EK
      EM(KS)=EM(KS)/EK
      EN(KS)=EN(KS)/EK
  603 CONTINUE
      WRITE(6,501) X,Y1,Z,PS1,EL(1),EM(1),EN(1),PS2,EL(2),EM(2),EN(2),PS
     13,EL(3),EM(3),EN(3)
  501 FORMAT(1H0,3F6.1,3(E13.4,3F6.2))
      GO TO 2
  200 R=SQRT((XX-YY)**2/4.&XY**2)
      PS1X=(XX&YY)/2.&R
      PS2X=(XX&YY)/2.-R
      DIFFX=2.*R
      DIRX=ARSIN(XY/R)*28.6
      IF(DIFFX.EQ.0.) DIRX=100000.
      R=SQRT((YY-ZZ)**2/4.&YZ**2)
      PS1Y=(YY&ZZ)/2.&R
      PS2Y=(YY&ZZ)/2.-R
      DIFFY=2.*R
      DIRY=ARSIN(YZ/R)*28.6
      IF(DIFFY.EQ.0.) DIRY=100000.
      R=SQRT((ZZ-XX)**2/4.&ZX**2)
      PS1Z=(ZZ&XX)/2.&R
      PS2Z=(ZZ&XX)/2.-R
      DIFFZ=2.*R
      DIRZ=ARSIN(ZX/R)*28.6
      IF(DIFFZ.EQ.0.) DIRZ=100000.
      WRITE(6,132) X,Y1,Z,DIFFX,DIRX,DIFFY,DIRY,DIFFZ,DIRZ
  132 FORMAT(1H03F6.1,3(E17.4,F6.1))
    2 CONTINUE
      IF(KCDE.EQ.1) GO TO 63
      IF(KCDE.EQ.2) GO TO 64
   20 CONTINUE
      END
```

A P P E N D I X I V

COMPUTER PROGRAMS FOR NUMERICAL INTEGRATION

Listings of the programs to carry out the two types of numerical integration described in Chapter 5 follow. The expanded form of the integrand is given by the variable PHIXU, which is defined in the programs.

```
C
C       TO SCLVE EQUATION 23 FOR J=K=1
C       USING THE MCNTE CARLC METHCD TC SOLVE THE INTEGRAL
C
      U1(X,Y,Z,C,R1,R2)=(T1/R1+1./R2+X**2/R1**3+T1*X**2/R2**3+2.*C*Z/R2*
     1*3*(1.-3.*X**2/R2**2)+T2/(R2+Z+C)*(1.-X**2/(R2*(R2+Z+C)))))/T3
      U2(X,Y,Z,C,R1,R2)=X*Y/T3*(1./R1**3+T1/R2**3-6.*C*Z/R2**5-T2/(R2*(R
     12+Z+C)**2))
      U3(X,Y,Z,C,R1,R2)=SQRT(X**2+Y**2)/T3*((Z-C)/R1**3+T1*(Z-C)/R2**3-T
     12/(R2*(R2+Z+C))+6.*C*Z*(Z+C)/R2**5)
      PHI1(X,Y,Z,C,R1,R2)=(-T1*(Z-C)/R1**3-(Z+C)/R2**3-3.*X**2*(Z-C)/R1*
     1*5-3.*T1*X**2*(Z+C)/R2**5-6.*C*Z*(Z+C)/R2**5+2.*C/R2**3+3C.*C*X**2
     2*Z*(Z+C)/R2**7-6.*C*X**2/R2**5-T2/(R2*(R2+Z+C))+T2*X**2*(2.*R2+Z+C
     6)/(R2**3*(R2+Z+C)**2)
     3                 +(Z-C)/R1**3+T1*(Z-C)/R2**3-6.*C*Z*(Z+C)/R2**5+T2
     4/(R2*(R2+Z+C))-X*(3.*(Z-C)*X/R1**5+3.*T1*(Z-C)*X/R2**5-30.*C*Z*(Z+
     5C)*X/R2**7+T2*X*(2.*R2+Z+C)/(R2**3*(R2+Z+C)**2)))/T3
      PHI2(X,Y,Z,C,R1,R2)=X/T3*(Y*(-3.*(Z-C)/R1**5-3.*T1*(Z+C)/R2**5-6.*
     1C/R2**5+30.*C*Z*(Z+C)/R2**7+T2*(2.*R2+Z+C)/(R2**3*(R2+Z+C)**2))-3.
     2*(Z-C)*Y/R1**5-3.*T1*Y*(Z-C)/R2**5+3C.*C*Z*(Z+C)*Y/R2**7-T2*Y*(2.*
     3R2+Z+C)/(R2**3*(R2+Z+C)**2))
      PHI3(X,Y,Z,C,R1,R2)=2./((1.-2.*PR)*T3)*((1.-PR)*X*(1./R1**3-3.*(Z-
     1C)**2/R1**5+T1/R2**3-3.*T1*(Z**2-C**2)/R2**5-6.*C*Z*(-5.*(Z+C)**2/
     2R2**7+1./R2**5)-6.*C*(Z+C)/R2**5-T2/R2**3)+PR*(-T1*X/R1**3-X/R2**3
     3+2.*X/R1**3-3.*X**3/R1**5+2.*T1*X/R2**3-3.*T1*X**3/R2**5-6.*C*Z*X/
     4R2**5-12.*C*Z*X/R2**5+30.*C*Z*X**3/R2**7-3.*T2*X/(R2*(R2+Z+C)**2)+
     5T2*X**3*(3.*R2+Z+C)/(R2**3*(R2+Z+C)**3)+X*(1./R1**3+T1/R2**3-6.*C*
     6Z/R2**5-T2/(R2*(R2+Z+C)**2)+Y*(-3.*Y/R1**5-3.*T1*Y/R2**5+3C.*C*Z*Y
     8/R2
     7**7+T2*Y*(3.*R2+Z+C)/(R2**3*(R2+Z+C)**3)))))
      PHIXU(A)=A*(U1(XD1,XD2,X3,Z3,R1,R2)*PHI1(Z1,Z2,Z3,Y3,S1,S2)+U2(XD1
     1,XD2,X3,Z3,R1,R2)*PHI2(Z1,Z2,Z3,Y3,S1,S2)+U3(XD1,XD2,X3,Z3,R1,R2)*
     2PHI3(Z1,Z2,Z3,Y3,S1,S2))
      READ(5,20) NORUN
   20 FORMAT(I5)
      DO21II=1,NORUN
      READ(5,10) G,PR,X1,X2,X3,Y3,RANGE,NOPTS,NOCYC,IRN1,IRN2,IRN3,H
   10 FORMAT(6F10.3/E10.4,5I10,F10.2)
      WRITE(6,11) X1,X2,X3,Y3,G,PR,RANGE,NCCYC,NOPTS,H,IRN1,IRN2,IRN3
   11 FORMAT('1EFFECT AT            X1 ='F8.2//19X,'X2 ='F8.2//19X,'X3 ='
     1F8.2//' DUE TO FCRCE 1 AT Y3 ='F8.2///' IN THE HALF SPACE  G ='
     2F8.2//19X,'PR ='F8.2/////' USING THE MONTE CARLO METHOD'///' WITH'
     311X,'RANGE ='F8.2//9X,'NO OF CYCLES ='I5//3X,'NO OF POINTS/CYCLE =
     4'I5//20X,'H ='F8.2//3X,'INITIAL RANDOM NOS ='I10//23X,I10//23X,
     5I10/////////'0INTEGRAL AFTER THIS CYCLE        AFTER ALL CYCLES SO
     6FAR'/)
      T1 = 3.-4.*PR
      T2 = 4.*(1.-PR)*(1.-2.*PR)
      T3=50.2656*(1.-PR)*G
      BIGSUM=0.
      DO2J=1,NCCYC
      SUM=0.
      DO1I=1,NOPTS
      CALL RAND(IRN1,IN1,RN1)
      Z1=(RN1-.5)*RANGE
      CALL RAND(IRN2,IN2,RN2)
```



```
      Z2=(RN2-.5)*RANGE
      CALL RAND(IRN3,IN3,RN3)
      Z3=RN3*RANGE
      IF(ABS(Z3-X3).GE.H) GO TO 4
      IF(ABS(Z2-X2).GE.H) GO TC 4
      IF(ABS(Z1-X1).LT.H) Z1=X1+H
    4 IF(ABS(Z3-Y3).GE.H) GO TC 5
      IF(ABS(Z2).GE.H) GO TO 5
      IF(ABS(Z1).LT.H) Z1=H
    5 XD1 = X1-Z1
      XD2 = X2-Z2
      R1 = SQRT(XD1**2+XD2**2+(X3-Z3)**2)
      R2 = SQRT(XD1**2+XD2**2+(X3+Z3)**2)
      S1 = SQRT(Z1**2+Z2**2+(Z3-Y3)**2)
      S2 = SQRT(Z1**2+Z2**2+(Z3+Y3)**2)
    1 SUM=SUM+PHIXU(G)
      ZN=NOPTS
      TOT=SUM/ZN*RANGE**3
      BIGSUM=BIGSUM+TOT
      ZJ=J
      BIGTOT=BIGSUM/ZJ
    2 WRITE(6,13) TOT,BIGTOT
   13 FORMAT('0'E22.6,E23.6)
      R1=SQRT(X1**2+X2**2+(X3-Y3)**2)
      R2=SQRT(X1**2+X2**2+(X3+Y3)**2)
      U11=U1(X1,X2,X3,Y3,R1,R2)
      V11=BIGTOT-Y3*U11
   21 WRITE(6,12) U11,V11
   12 FORMAT(//////' U11 ='E14.6///' V11 ='E14.6///////////)
      WRITE(6,14)IRN1,IRN2,IRN3
   14 FORMAT('0'I80/'0'I80/'0'I80)
      END
```

MODULE RAND

```
      SUBROUTINE RAND(IX,IY,YFL)
      IY=IX*65539
      IF(IY)5,6,6
    5 IY=IY+2147483647+1
    6 YFL=IY
      IX=IY
      YFL=YFL*.4656613E-9
      RETURN
      END
```

```
C       TC SCLVE EQUATICN 23 FUR J=K=1
C       USING THE TRAPEZCIDAL RULE TC EVALUATE THE INTEGRAL
C
        U1(X,Y,Z,C,R1,R2)=(T1/R1+1./R2+X**2/R1**3+T1*X**2/R2**3+2.*C*Z/R2*
       1*3*(1.-3.*X**2/R2**2)+T2/(R2+Z+C)*(1.-X**2/(R2*(R2+Z+C))))/T3
        U2(X,Y,Z,C,R1,R2)=X*Y/T3*(1./R1**3+T1/R2**3-6.*C*Z/R2**5-T2/(R2*(R
       12+Z+C)**2))
        U3(X,Y,Z,C,R1,R2)=SQRT(X**2+Y**2)/T3*((Z-C)/R1**3+T1*(Z-C)/R2**3-T
       12/(R2*(R2+Z+C))+6.*C*Z*(Z+C)/R2**5)
        PHI1(X,Y,Z,C,R1,R2)=(-T1*(Z-C)/R1**3-(Z+C)/R2**3-3.*X**2*(Z-C)/R1*
       1*5-3.*T1*X**2*(Z+C)/R2**5-6.*C*Z*(Z+C)/R2**5+2.*C/R2**3+3C.*C*X**2
       2*Z*(Z+C)/R2**7-6.*C*X**2/R2**5-T2/(R2*(R2+Z+C))+T2*X**2*(2.*R2+Z+C
       6)/(R2**3*(R2+Z+C)**2)
       3                    +(Z-C)/R1**3+T1*(Z-C)/R2**3-6.*C*Z*(Z+C)/R2**5+T2
       4/(R2*(R2+Z+C))-X*(3.*(Z-C)*X/R1**5+3.*T1*(Z-C)*X/R2**5-30.*C*Z*(Z+
       5C)*X/R2**7+T2*X*(2.*R2+Z+C)/(R2**3*(R2+Z+C)**2)))/T3
        PHI2(X,Y,Z,C,R1,R2)=X/T3*(Y*(-3.*(Z-C)/R1**5-3.*T1*(Z+C)/R2**5-6.*
       1C/R2**5+30.*C*Z*(Z+C)/R2**7+T2*(2.*R2+Z+C)/(R2**3*(R2+Z+C)**2))-3.
       2*(Z-C)*Y/R1**5-3.*T1*Y*(Z-C)/R2**5+30.*C*Z*(Z+C)*Y/R2**7-T2*Y*(2.*
       3R2+Z+C)/(R2**3*(R2+Z+C)**2))
        PHI3(X,Y,Z,C,R1,R2)=2./((1.-2.*PR)*T3)*((1.-PR)*X*(1./R1**3-3.*(Z-
       1C)**2/R1**5+T1/R2**3-3.*T1*(Z**2-C**2)/R2**5-6.*C*Z*(-5.*(Z+C)**2/
       2R2**7+1./R2**5)-6.*C*(Z+C)/R2**5-T2/R2**3)+PR*(-T1*X/R1**3-X/R2**3
       3+2.*X/R1**3-3.*X**3/R1**5+2.*T1*X/R2**3-3.*T1*X**3/R2**5-6.*C*Z*X/
       4R2**5-12.*C*Z*X/R2**5+30.*C*Z*X**3/R2**7-3.*T2*X/(R2*(R2+Z+C)**2)+
       5T2*X**3*(3.*R2+Z+C)/(R2**3*(R2+Z+C)**3)+X*(1./R1**3+T1/R2**3-6.*C*
       6Z/R2**5-T2/(R2*(R2+Z+C)**2)+Y*(-3.*Y/R1**5-3.*T1*Y/R2**5+30.*C*Z*Y
       8/R2
       7**7+T2*Y*(3.*R2+Z+C)/(R2**3*(R2+Z+C)**3)))))
        PHIXU(A)=A*(U1(XD1,XD2,X3,Z3,R1,R2)*PHI1(Z1,Z2,Z3,Y3,S1,S2))
        READ(5,20) NCRUN
     20 FORMAT(I5)
        DO21II=1,NCRUN
        READ(5,10) G,PR,X1,X2,X3,Y3,RANGE,NOPTS,H,PSTEP
     10 FORMAT(6F10.3/E10.4,I10,2F10.2)
        WRITE(6,11) X1,X2,X3,Y3,G,PR,RANGE,NCPTS,H,PSTEP
     11 FORMAT('1EFFECT AT          X1 ='F8.2//19X,'X2 ='F8.2//19X,'X3 ='
       1F8.2//' DUE TO FORCE 1 AT Y3 ='F8.2///' IN THE HALF SPACE  G ='
       2F8.2//19X,'PR ='F8.2/////' USING THE TRAPEZOIDAL RULE'///' WITH'
       311X,'RANGE ='F8.2//9X,'NC CF PCINTS ='I5//20X,'H ='F8.2//16X,
       4'PSTEP ='F8.2//////////)
        T1 = 3.-4.*PR
        T2 = 4.*(1.-PR)*(1.-2.*PR)
        T3=50.2656*(1.-PR)*G
        ZN=NCPTS
        STEP=RANGE/ZN
     30 SUM=C.
        DO 1 I=1,NOPTS
        ZI=I
        Z1=-RANGE/2.+(ZI-PSTEP)*STEP
        DO1J=1,NCPTS
        ZJ=J
        Z2=-RANGE/2.+(ZJ-PSTEP)*STEP
        DO1K=1,NCPTS
        ZK=K
        Z3=(ZK-PSTEP)*STEP
```



```
        IF(ABS(Z3-X3).GE.H) GO TC 4
        IF(ABS(Z2-X2).GE.H) GO TC 4
        IF(ABS(Z1-X1).LT.H) Z1=X1+H
      4 IF(ABS(Z3-Y3).GE.H) GO TO 5
        IF(ABS(Z2).GE.H) GC TO 5
        IF(ABS(Z1).LT.H) Z1=H
      5 XD1 = X1-Z1
        XD2 = X2-Z2
        R1 = SQRT(XD1**2+XD2**2+(X3-Z3)**2)
        R2 = SQRT(XD1**2+XD2**2+(X3+Z3)**2)
        S1 = SQRT(Z1**2+Z2**2+(Z3-Y3)**2)
        S2 = SQRT(Z1**2+Z2**2+(Z3+Y3)**2)
      1 SUM=SUM+PHIXU(G)
        SUM=SUM*RANGE**3/ZN**3
        WRITE(6,13) SUM
     13 FORMAT(13X,'INTEGRAL ='E14.6/)
        R1=SQRT(X1**2+X2**2+(X3-Y3)**2)
        R2=SQRT(X1**2+X2**2+(X3+Y3)**2)
        U11=U1(X1,X2,X3,Y3,R1,R2)
        V11=SUM-Y3*U11
     21 WRITE(6,12) U11,V11
     12 FORMAT(18X,'U11 ='E14.6//18X,'V11 ='E14.6////////////)
        END
```

APPENDIX V

FIRST ATTEMPT AT CONSIDERATION OF A LAYERED HALF-SPACE

V.1 INTRODUCTION

It was mentioned in section 6.2.1 that a solution had been developed using an initial basic assumption described in that section. This initial assumption concerning the behaviour of a layered elastic half-space was later discarded because an improved basic assumption was adopted. The theory which was developed from the initial assumption is described here for interest.

V.2 INITIAL THEORY

V.2.1 Introduction

The assumption is contained in section 6.2.1.1 and is not repeated here. The initial part of the analysis which is concerned with a single pile element is similar to that described in Chapter 3. Additional terms occur because the pile element will in general have a non-zero moment M_2 and shear V_2 acting at its lower end in addition to M_1 (replacing M) and V_1 (replacing V) at its upper end. z_i and z_j must now be measured from the top of the pile element, n refers to the number of forces acting on the element, and E, I, and L all refer to the single element under consideration rather than to the complete pile.

The following sections correspond to those in

Chapter 3, starting at section 3.2.2, and should be read in conjunction with that chapter.

V.2.2. Pile Equations

As before,

$$u_i = \frac{L - z_i}{L} d_1 + \frac{z_i}{L} d_2 - w_i \quad .$$

The expression for w_i contains an extra term:

$$w_i = \frac{M_1 z_i}{6EIL} (2L - z_i)(L - z_i) + \frac{M_2 z_i}{6EIL} \ (1 - z_i)\ (L + z_i)$$

$$- \sum_{j=1}^{n} \frac{P_j z_i (L - z_j)}{6EIL} \ (z_i^{\ 2} + z_j^{\ 2} - 2Lz_j)$$

$$+ \left[\frac{P_j z_j}{6EIL} (z_i - z_j)^3 \right]_{z_i > z_j \text{ only}}$$

That is,

$$w_i = a_{ik} M_k + b_{ij} P_j$$

where k takes the values 1, 2 and j takes the values 1 - n.

The n equations for pile displacement are thus

$$u_i = \frac{L-z_i}{L} d_1 + \frac{z_i}{L} d_2 - a_{ik} M_k - b_{ij} P_j \qquad \text{(V.1)}$$

V.2.3 Soil Equations

Equation V.2 may be taken as identical to equation 3.2, where G and μ are the soil properties corresponding to the element under consideration and z is in this case measured from the soil surface. Thus we may write, corresponding to equation (3.3),

$$u_i = c_{ij} P_j \qquad \text{(V.3)}$$

V.2.4 Equilibrium equations

The equilibrium equations now become

$$\left.\begin{aligned} \sum_{j=1}^{n} P_j &= V_1 + V_2 \\ \text{and} \qquad P_j z_j &= M_2 - M_1 + V_2 L \end{aligned}\right\} \qquad (V.4)$$

V.2.5 Solution of equations

Combining (V.1) and (V.3) to eliminate u_i,

$$\frac{L-z_i}{L} d_1 + \frac{z_i}{L} d_2 - a_{ik} M_k - b_{ij} P_j - c_{ij} P_j = 0$$

or $$r_{ik} d_k + s_{ij} P_j = a_{ik} M_k \qquad (V.5)$$

Equation (V.6) will now be identical to equation (3.6), except that the submatrices A and B contain extra terms.

We must now depart from the theory of Chapter 3, for M_k and V_k are in general not known, where previously we could in effect write $M_1 = M$, $V_1 = V$, and $M_2 = V_2 = 0$.

V.3 EXTENDED THEORY

It is necessary now to find the end actions on each pile element.

If the rotation at the upper end of the pile element is s_1 and at the lower end s_2 and positive rotation is clockwise, then

$$s_1 = \frac{d_1-d_2}{L} + \frac{1}{6EIL}\left[2M_1L^2+M_2L^2+M_p\,(2(L-z_j)^2+z_j(3L-2z_j))\right]$$

and

$$s_2 = \frac{d_1 - d_2}{L} - \frac{1}{6EIL}\left[M_1 L^2 + 2M_2 L^2 + M_p((L-z_j)(2z_j+L)+2z_j^2) \right]$$

where $$M_p = \frac{P_j(L-z_j)z_j}{L} \qquad (V.7)$$

Now for the internal equilibrium of the pile

$$M_{1,\ell} = M_{2,\ell-1}$$

and $$V_{1,\ell} = -V_{2,\ell-1} \quad , \qquad (V.8)$$

where the second subscript gives the number of a pile element (the number of the upper boundary). For the compatibility of the pile,

$$d_{1,\ell} = d_{2,\ell-1}$$

and $$s_{1,\ell} = s_{2,\ell-1} \quad . \qquad (V.9)$$

The actions and displacements may be related by the equations

$$\begin{Bmatrix} d_1 \\ s_1 \\ d_2 \\ s_2 \end{Bmatrix}_\ell = \begin{bmatrix} & & \\ & \underset{(4\times 4)}{R} & \\ & & \end{bmatrix}_\ell \begin{Bmatrix} M_1 \\ V_1 \\ M_2 \\ V_2 \end{Bmatrix}_\ell \qquad (V.10)$$

where R is a stiffness matrix the terms of which can be obtained from equations (V.6) and (V.7). Two new quantities are now introduced and defined by the equations

$$\Delta_\ell = d_{2,\ell-1} - d_{1,\ell}$$

and $$\theta_\ell = s_{2,\ell-1} - s_{1,\ell} \qquad (V.11)$$

Expanding from (V.9),

$$\Delta_\ell = (r_{31}\, M_1 + r_{32}\, V_1 + r_{33}\, M_2 + r_{34}\, V_2)_{\ell-1}$$

$$- (r_{11}\, M_1 + r_{12}\, V_1 + r_{13}\, M_2 + r_{14}\, V_2)_\ell$$

and $$\theta_\ell = (r_{41}\, M_1 + r_{42}\, V_1 + r_{43}\, M_2 + r_{44}\, V_2)_{\ell-1}$$

$$- (r_{21}\, M_1 + r_{22}\, V_1 + r_{23}\, M_2 + r_{24}\, V_2)_\ell$$

These may be simplified by writing

$$M_{1,\ell} = M_{2,\ell-1} = M_\ell$$

and $$-V_{1,\ell} = V_{2,\ell-1} = V_\ell$$

to give

$$\Delta_\ell = r_{31,\ell-1}\, M_{\ell-1} - r_{32,\ell-1}\, V_{\ell-1} + (r_{33,\ell-1} - r_{11,\ell})M_\ell$$

$$+ (r_{34,\ell-1} + r_{12,\ell})V_\ell - r_{13,\ell}\, M_{\ell+1} - r_{14,\ell}\, V_{\ell+1}$$

and

$$\theta_\ell = r_{41,\ell-1}\, M_{\ell-1} - r_{42,\ell-1}\, V_{\ell-1} + (r_{43,\ell-1} - r_{21,\ell})M_\ell$$

$$+ (r_{44,\ell-1} + r_{22,\ell})V_\ell - r_{23,\ell}\, M_{\ell+1} - r_{24,\ell}\, V_{\ell+1} ,$$

or $$\Delta_\ell = t_{11}\, M_{\ell-1} + t_{12}\, V_{\ell-1} + t_{13} M_\ell + t_{14} V_\ell + t_{15} M_{\ell+1}\, t_{16} V_{\ell+1}$$

and $$= t_{21}\, M_{\ell-1} + t_{22}\, V_{\ell-1} + t_{23} M_\ell + t_{24} V_\ell + t_{25} M_{\ell+1} + t_{26} V_{\ell+1} .$$

As the quantities $d_{2,o}$ and $s_{2,o}$ do not exist, these equations are meaningless for $\ell = 1$, but they are meaningful for $\ell = 2$ to m, where m is the number of soil layers. The equation set can then be written as (for the case $m = 4$):

$$\begin{Bmatrix} \Delta_2 \\ \theta_2 \\ \Delta_3 \\ \theta_3 \end{Bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} & \cdot & \cdot \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} & \cdot & \cdot \\ \cdot & \cdot & t_{33} & t_{34} & t_{35} & t_{36} & t_{37} & t_{38} \\ \cdot & \cdot & t_{43} & t_{44} & t_{45} & t_{46} & t_{47} & t_{48} \end{bmatrix} \begin{Bmatrix} M_1 \\ V_1 \\ M_2 \\ V_2 \\ M_3 \\ V_3 \\ M_4 \\ V_4 \end{Bmatrix} \qquad (V.12)$$

$2(m-1)x1 \qquad 2(m-1)x2(m+1) \qquad 2(m+1)x1$

where M_1 and V_1 are the actions at the top end of the pile and M_{m+1} and V_{m+1} act at the bottom end.

Now equations (V.9) require

$$\Delta_\ell = \theta_\ell = 0 \quad ,$$

which means that trivial solution,

$$M_\ell = V_\ell = 0 \quad ,$$

is obtained for equations (V.12).

A non-trivial solution is required, and this is only possible if Δ_ℓ and θ_ℓ are in general non-zero, which will be the case if any actions other than the correct ones are applied at the ends of the pile elements. We shall thus apply any actions at the ends of the pile elements. They must satisfy equilibrium, as in equation (V.8), but pile continuity will not be preserved and Δ_ℓ and θ_ℓ as determined from equations (V.6), (V.7) and (V.11) will in general have non-zero values.

If now $-\Delta_\ell$ and $-\theta_\ell$ are written for Δ_ℓ and θ_ℓ in equations (V.12), it can be seen that M_ℓ and V_ℓ will be the additional actions required for pile compatibility.

Further, if the chosen values for M_1, V_1, M_{m+1} and V_{m+1} are

$$M_1 = \text{applied moment } M,$$
$$V_1 = \text{applied shear } V,$$
$$M_{m+1} = 0 \text{ , and}$$
$$V_{m+1} = 0$$

then the additional actions will in these four cases be zero. The first two and last two columns of matrix T in equation (V.12) may now be omitted as well as the above zero quantities for which they are the coefficients, and equations (V.12) rewritten as

$$\begin{Bmatrix} -\Delta_2 \\ -\theta_2 \\ -\Delta_3 \\ -\theta_3 \end{Bmatrix} = \begin{bmatrix} t_{13} & t_{14} & t_{15} & t_{16} \\ t_{23} & t_{24} & t_{25} & t_{26} \\ t_{33} & t_{34} & t_{35} & t_{36} \\ t_{43} & t_{44} & t_{45} & t_{46} \end{bmatrix} \begin{Bmatrix} M_2 \\ V_2 \\ M_3 \\ V_3 \end{Bmatrix}$$

or, renaming the variables, and giving the general case in terms of the boundary number ℓ,

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ \\ \\ d_{2\ell-3} \\ d_{2\ell-2} \\ \\ \\ d_{2m-3} \\ d_{2m-2} \end{Bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & \cdot & \cdot & \cdot & \cdot & \cdot \\ t_{21} & t_{22} & t_{23} & t_{24} & \cdot & \cdot & \cdot & \cdot & \cdot \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} & t_{36} & \cdot & \cdot & \cdot \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} & t_{46} & \cdot & \cdot & \cdot \\ \cdot & \cdot & t_{53} & t_{54} & t_{55} & t_{56} & t_{57} & t_{58} & \cdot \\ \cdot & \cdot & t_{63} & t_{64} & t_{65} & t_{66} & t_{67} & t_{68} & \cdot \\ \\ \\ & & & t_{2\ell-3,2\ell-5} - t_{2\ell-3,2\ell} & & & & & \\ & & & t_{2\ell-2,2\ell-5} - t_{2\ell-2,2\ell} & & & & & \\ \\ \\ & & & & & & & & t_{2m-3,2m-5} - t_{2m-3,2m-2} \\ & & & & & & & & t_{2m-2,2m-5} - t_{2m-2,2m-2} \end{bmatrix} \begin{Bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ \\ \\ m_{2\ell-3} \\ m_{2\ell-2} \\ \\ \\ m_{2m-3} \\ m_{2m-2} \end{Bmatrix}$$

that is,

$$D = T . M$$

and $$M = T^{-1} D \qquad (V.13)$$

The additional actions may now be evaluated, added to those guessed and used to write matrices A and B in equations (V.6).

V.4 SUMMARY OF PROCEDURE

1. Set up equations (V.6) for each pile element.
2. Use these to evaluate matrix R in (V.10) for each element.
3. Using $M_1 = M$, $V_1 = V$, $M_{m+1} = V_{m+1} = 0$, and all other actions as say $M = V = 1$, evaluate matrices A_ℓ and B_ℓ in (V.6)
4. Use (V.6), (V.7) and (V.11) to evaluate matrix D in (V.13).
5. Assemble and invert matrix T in (V.12).
6. Solve equations (V.13) for M.
7. Add M to actions from step 3 and use to re-evaluate A_ℓ and B_ℓ.
8. Solve equations (V.6) for P and D.
9. Solve equations (V.2) for deflected shape.
10. Calculate other quantities, for example bending moment and shear force distributions.